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A FIRST COURSE IN ALGEBRA

BY

WEBSTER WELLS, S.B.

PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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PREFACE

In the preparation of this text the author acknowledges joint-authorship with Robert L. Short.

This book meets the demand that the pupil be given an elementary algebra containing no more than can be accomplished in the time allotted to the subject. It is not intended for a complete course, but gives the student a good working knowledge of the subject through simultaneous quadratics. It should be followed by a second course by those intending to pursue the study of higher mathematical subjects. This book is sufficient preparation for geometry, and the frequent introduction of geometric ideas and geometric problems not only prepares for geometry but also makes that subject attractive to the learner.

This text is as brief as the algebra of years ago, and yet contains all that is good in modern mathematical thought. Attention is called to the introduction of graphical methods through simple horizontal and vertical measurements (Exercise 4, Exercise 41, problems 28–30). This procedure makes the transition to Cartesian coördinates a natural one. Teachers will find that the color scheme recommended in graphs will greatly aid the student in connecting related data. Pedagogical advantage is gained through the combining of related and reverse processes. (Chapters III, VII, X, XII, XIII.)

The use of the fractional exponent in operations involving surds is recommended, thereby avoiding confusion, since the four fundamental laws and the exponential laws of Multiplication, Division, Involution, and Evolution, are the only ones involved. The complete index will be found helpful to both pupil and teacher. No attempt is made toward technical

definition. Definitions for the beginner must be explanatory and descriptive. The lists of queries will aid in fixing both definitions and principles.

The authors thank the many teachers of mathematics who have made this book better and have brought it close to actual class-room conditions by their timely criticism and suggestion.

WEBSTER WELLS.

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ALGEBRA

I. DEFINITIONS AND NOTATION

SYMBOLS REPRESENTING NUMBERS

1. In **Algebra** the symbols usually employed to represent numbers are the *Arabic numerals* and the *letters of the alphabet.*

The numerals represent known or determinate numbers.

The letters represent numbers which may have any values whatever, or numbers whose values are to be found.

EQUATIONS

2. The **Sign of Equality**, =, is read "*equals.*"

Thus, $a=b$ signifies that the number a equals the number b.

3. An **Equation** is an expression of equality.

The *first member* of an equation is the number to the left of the sign of equality, and the *second member* is the number to the right of that sign; thus, in the equation $2x-3=5$, the first member is $2x-3$, and the second member 5.

AXIOMS

4. An **Axiom** is a statement which is assumed as self-evident. Algebraic operations of finite numbers are based in part on the following axioms:

1. Any number equals itself.
2. Any number equals the sum of all its parts.
3. Any number is greater than any of its parts.
4. Two numbers which are equal to the same number or to equal numbers, are equal.

5. If the same number, or equal numbers, be added to equal numbers, the resulting numbers will be equal.

6. If the same number, or equal numbers, be subtracted from equal numbers, the resulting numbers will be equal.

7. If equal numbers be multiplied by the same number, or equal numbers, the resulting numbers will be equal.

8. If equal numbers be divided by the same number, or equal numbers, the resulting numbers will be equal. Numbers cannot be divided by the number 0.

SOLUTION OF PROBLEMS BY ALGEBRAIC METHODS

5. The following examples illustrate some uses of algebraic symbols:

1. The sum of two numbers is 30, and the greater exceeds the less by 4; what are the numbers?

We will represent the less number by x.

Then the greater will be represented by $x+4$.

By the conditions of the problem, the sum of the *less* number and the *greater* is 30; this is stated in algebraic language as follows:

$$x+x+4=30. \qquad (1)$$

$$x+x=2x.$$

Therefore, $$2x+4=30.$$

The members of this equation, $2x+4$ and 30, are equal numbers; if from each of them we subtract the number 4, the resulting numbers will be equal (Ax. 6, § 4).

Therefore, $$2x=30-4, \text{ or } 2x=26.$$

Dividing the equal numbers $2x$ and 26 by 2 (Ax. 8, § 4), we have

$$x=13.$$

Hence, the less number is 13, and the greater is $13+4$, or 17.

The written work will stand as follows:

Let	$x=$ the less number.
Then,	$x+4=$ the greater number.
By the conditions,	$x+x+4=30$, or $2x+4=30$.
Whence,	$2x=26$.
Dividing by 2,	$x=13$, the less number.
Whence,	$x+4=17$, the greater number.

2. The sum of the ages of A and B is 109 years, and A is 13 years younger than B; find their ages.

Let n represent the number of years in B's age.

Then, $n-13$ will represent the number of years in A's age.

By the conditions of the problem, the sum of the ages of A and B is 109 years.

Whence, $n-13+n=109$, or $2n-13=109$.

Adding 13 to both members (Ax. 5, § 4),

$$2n=122.$$

Dividing by 2, $n=61$, the number of years in B's age.

And, $n-13=48$, the number of years in A's age.

The written work will stand as follows:

Let $n=$ the number of years in B's age.
Then, $n-13=$ the number of years in A's age.
By the conditions, $n-13+n=109$, or $2n-13=109$.
Whence, $2n=122$.
Dividing by 2, $n=61$, the number of years in B's age.
Therefore, $n-13=48$, the number of years in A's age.

In Ex. 2, we do not say "let n represent B's *age*," but "let n represent the *number of years* in B's age."

3. A, B, and C together earn $66. A's share is one-half as much as B's, and C's is 3 times as much as A's. How much has each?

Let $x=$ the number of dollars A has.
Then, $2x=$ the number of dollars B has.
and $3x=$ the number of dollars C has.

By the conditions, $x+2x+3x=66$.

But the sum of x, twice x, and 3 times x is 6 times x, or $6x$.

Whence, $6x=66$.
Dividing by 6, $x=11$, the number of dollars A has.
Whence, $2x=22$, the number of dollars B has,
and $3x=33$, the number of dollars C has.

(By letting x represent the number of dollars A has, in Ex. 3, we avoid fractions.)

EXERCISE 1

Write the following in algebraic symbols:

1. One number is 4 more than another. What is their sum? (Hint: Let x = the smaller number.)

2. There are three numbers such that the second is twice the first, and the third thrice the first. What is their sum?

3. The sum of two numbers is 20 and one of the numbers is x. What is the other number?

4. If one number is 4 times another, what is their difference?

5. Write: the sum of 5 times a certain number and 3 times the number, divided by 3.

6. The sum of two numbers is a and one of the numbers is b. What is the other?

7. The greater of two numbers is 8 times the less, and exceeds it by 49; find the numbers.

8. The sum of the ages of A and B is 119 years, and A is 17 years older than B; find their ages.

9. Divide $74 between A and B so that A may receive $48 more than B.

10. Divide $108 between A and B so that A may receive 5 times as much as B.

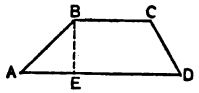
11. Divide 91 into two parts such that the smaller shall be one-sixth of the greater.

12. A man travels 112 miles by train and steamer; he goes by train 54 miles farther than by steamer. How many miles does he travel in each way?

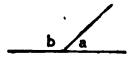
13. The sum of three numbers is 69; the first is 14 greater than the second, and 28 greater than the third. Find the numbers.

14. The area of a trapezoid is equal to the product of one-half the sum of the parallel sides and the altitude. In the trapezoid $ABCD$, AD is 8 more than BC, EB is 6, and the

area of the trapezoid is 54. Find the length of BC and of AD. Is the drawing correct?



15. The sum of two angles a and b is 180°. The angle b is three times as great as the angle a. Find the number of degrees in each angle.

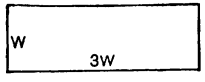


16. Divide \$6.75 between A and B so that A may receive one-fourth as much as B.

17. A man has \$2. After losing a certain sum, he finds that he has left 20 cents more than 3 times the sum which he lost. How much did he lose?

18. A, B, and C in partnership gain \$140; A is to have 4 times as much as B, and C as much as A and B together. Find the share of each.

19. One side of a rectangle is thrice the side adjacent to it. The opposite sides are equal, and the sum of the sides is 24 inches; find the length and breadth.



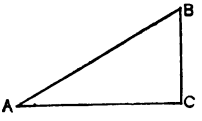
20. At an election two candidates, A and B, had together 653 votes, and A was beaten by 395 votes. How many did each receive?

21. A field is 7 times as long as it is wide, and the distance around it is 240 feet. Find its dimensions.

22. My horse, carriage, and harness are worth together \$325. The horse is worth 6 times as much as the harness, and the carriage is worth \$65 more than the horse. How much is each worth?

23. The sum of three numbers is 87; the third number is one-eighth of the first, and the second number 15 less than the first. Find the numbers.

24. The sum of the three angles of a triangle is always 180°. In a triangle ABC, angle B is 30° larger than angle A, and angle C is 30° larger than angle B. Find the number of degrees in each angle.



25. The sum of the ages of A, B, and C is 110 years; B's age exceeds twice C's by 12 years, and A is 9 years younger than B. Find their ages.

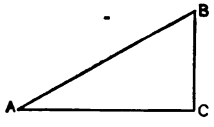
26. A pole 77 feet long is painted red, white, and black; the red is one-fifth of the white, and the black 21 feet more than the red. How many feet are there of each color?

27. Divide 70 into three parts such that the third part shall be one-fifth of the first, and one-fourth of the second.

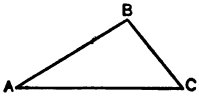
28. In an algebra class of 27 pupils there are twice as many girls as boys. How many girls in the class?

29. A, B, and C have together \$22.50; B has \$1.50 more than A, and C has \$8 less than twice the amount that A has. How much has each?

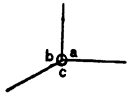
30. In a triangle, ABC, angle C is 90°, angle B is twice angle A. The sum of the three angles is 180°. Find the angles A and B.



31. The sum of the three sides of a triangle, ABC, is 35 feet. Side AB is 4 feet more than side BC and side AC is 7 feet more than side BC. Find the length of AB and AC.



32. Three straight lines are drawn from a point O forming the angles a, b, and c. b is 30° larger than a; c is 30° larger than b. The sum of the three angles is 360°. Find the number of degrees in each angle.



DEFINITIONS

6. The continued product of a number by itself any number of times is called a **Power** of that number.

An **Exponent** is a number written at the right of, and above another number called the **Base**, to indicate what power of the latter is to be taken; thus,

a^2, read "*a square*," or "*a second power*," denotes $a \times a$;
a^3, read "*a cube*," or "*a third power*," denotes $a \times a \times a$;
a^4, read "*a fourth*," "*a fourth power*," or "*a exponent* 4," denotes $a \times a \times a \times a$, etc.

The meaning of exponent will be extended in Chap. XIII.

If no exponent is expressed, the *first* power is understood. Thus, a is the same as a^1.

7. Symbols of Aggregation.

The *parentheses* (), the *brackets* [], the *braces* { }, and the *vinculum* ——, indicate that the numbers enclosed by them are to be taken collectively; thus,

$$(a+b)\times c,\ [a+b]\times c,\ \{a+b\}\times c, \text{ and } \overline{a+b}\times c$$

all indicate that the result obtained by adding b to a is to be multiplied by c.

If an expression involves *parentheses*, the operations indicated *within* the parentheses must be performed first.

EXERCISE 2

Write the following in symbols:

1. The result of subtracting 6 times n from 5 times m.

2. Three times the product of the eighth power of m and the ninth power of n.

3. The quotient of the sum of a and b divided by the sum of c and d.

What operations are signified by the following?

4. $2x^5y^6$.

5. $m(x-y)$.

6. $\frac{cx}{mh}$.

7. $3+(y-z)$.

8. $3-(y+z)$.

9. $(m-n)^4$.

10. $\frac{a}{b}-\frac{c}{d}$.

11. $\left(\frac{x+z}{x-z}\right)^2$.

12. $(2a+b)(4c-5d)$.

13. $\left(\frac{1}{x}+\frac{1}{y}\right)z^4$.

Write the following in symbols:

14. The product of $3x+y$ and z^2.

15. The result of subtracting $y-z$ from x.

16. The product of $a-b$ and $c-d$.

17. The result of adding the quotient of m by n, and the quotient of x by y.

18. The square of $m+n$.

19. The cube of $a-b+c$.

20. Translate into English $\frac{a+b}{2}$; $\frac{a-b}{2}$.

21. In the above example is a a number? What value has it? If a were 5 and b were 3, what would be the value of the fraction?

22. Translate into English $\frac{x+y}{x-y}$.

23. In example 22, if x is 7 and y is 5, find the fraction.

ALGEBRAIC EXPRESSIONS

8. An **Algebraic Expression**, or simply an **Expression**, is a number expressed in algebraic symbols; as,

$$2,\quad a,\quad \text{or}\quad 2x^2-3ab+5.$$

9. The **Numerical Value** of an expression is the result obtained by substituting particular numerical values for the letters involved in it, and performing the operations indicated.

1. Find the numerical value of the expression

$$4a+\frac{6c}{b}-d^3,$$

when $a=4$, $b=3$, $c=5$, and $d=2$.

We have, $4a+\frac{6c}{b}-d^3=4\times4+\frac{6\times5}{3}-2^3=16+10-8=18.$

2. Find the numerical value, when $a=9$, $b=7$, and $c=4$, of

$$(a-b)(b+c)-\frac{a+b}{b-c}$$

First perform the operations indicated in parentheses.
We have, $a-b=2$, $b+c=11$, $a+b=16$, and $b-c=3$.
Then the numerical value of the expression is

$$2\times 11-\frac{16}{3}=22-\frac{16}{3}=\frac{50}{3}.$$

EXERCISE 3

Find the numerical values of the following when $a=6$, $b=3$, $c=4$, $d=5$, $m=3$, and $n=2$:

1. a^2b-cd^2.
2. $2\,abcd$.
3. $3\,ab+4\,bc-5\,cd$.
4. a^mb^n.
5. a^b+b^a.
6. $\dfrac{a}{bc}+\dfrac{c}{ad}$.
7. $\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{d}$.
8. $\dfrac{15\,c^m}{28\,d^m}$.
9. $\dfrac{a}{b}-\dfrac{b}{c}+\dfrac{c}{d}$.
10. $\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}-\dfrac{1}{d}$.
11. $\dfrac{5\,c^2d}{a^3}-\dfrac{cd^2}{2\,b^3}$.
12. $\dfrac{b^2}{a^2}+\dfrac{c^2}{b^2}-\dfrac{d^2}{c^2}$.

Find the numerical values of the following when $a=5$, $b=2$, $c=3$, and $d=4$:

13. $\left(\dfrac{2\,a+d}{2\,b+c}\right)^2$.
14. $(a^2-b^2-d^2)^3$.
15. $5\,a^2(a-b)-2\,b^3(c+d)$.
16. $8(a-b)^2+3(c+d)^2$.
17. $(a-b)^2+(2\,a-3\,b)^2-(b+c)^2$.
18. $(2\,a-b-c+d)(2\,a+b+c-d)$.
19. $\dfrac{8\,a+3\,b-6\,c}{9\,a-4\,b-3\,c}$.
20. $\dfrac{a-b}{a+b}+\dfrac{a-c}{a+c}+\dfrac{a-d}{a+d}$.

Find the numerical values of the following when $a=\frac{3}{4}$, $b=\frac{5}{2}$, $c=\frac{1}{3}$, and $x=4$:

21. $\dfrac{a+c}{a-c}-\dfrac{a-c}{a+c}$.
22. $\dfrac{8\,a+6\,b-15\,c}{16\,a+10\,b+9\,c}$.
23. $x^3+(2\,a+3\,b)x^2-(5\,a-4\,c)x+\frac{8}{3}\,abc$.
24. $x^3-\dfrac{2}{a}+\dfrac{3}{b}x^2+\dfrac{6}{a}-\dfrac{5}{b}+\dfrac{4}{c}x-\dfrac{13}{abc}$.

II. POSITIVE AND NEGATIVE NUMBERS

10. In financial transactions, we may have *assets* or *liabilities*, and *gains* or *losses;* we may have motion along a straight line in a certain direction, or in the *opposite* direction; etc. Taken in pairs, these ideas have opposite meanings or opposite sense.

In each of these cases, the effect of combining with a magnitude of a certain kind another of the opposite kind is to diminish the former, destroy it, or reverse its state.

Thus, if to a certain amount of asset we add a certain amount of liability, the asset is diminished, destroyed, or changed into liability.

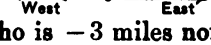
11. The signs + and −, besides denoting addition and subtraction, are also used, in Algebra, to distinguish between the opposite states of magnitudes like those of § 10.

Thus, we may indicate *assets* by the sign +, and *liabilities* by the sign −; for example, the statement that a man's assets are −$100 means that he has liabilities to the amount of $100.

EXERCISE 4

1. If a man has assets of $400, and liabilities of $600, how much is he worth?

2. If gains be taken as positive, and losses as negative, what does a gain of −$100 mean?

3. In what position is a man who is −50 feet east of a certain point P? See figure. 

4. In what position is a man who is −3 miles north of a certain place? Draw the figure showing this.

5. How many miles north of a certain place is a man who goes 5 miles north, and then 9 miles south? Draw figure.

12. Positive and Negative Numbers.

If the positive and negative states of any concrete magnitude be expressed *without reference to the unit*, the results are called *positive* and *negative numbers*, respectively.

Thus, in $+\$5$ and $-\$3$, $+5$ is a positive number, and -3 is a negative number.

For this reason the sign $+$ is called the *positive* sign, and the sign $-$ the *negative* sign.

If no sign is expressed, the number is understood to be positive; thus, 5 is the same as $+5$.

The negative sign must never be omitted before a negative number.

13. **The Absolute Value** of a number is the number taken independently of the sign affecting it.

Thus, the absolute value of -3 is 3.

ADDITION OF POSITIVE AND NEGATIVE NUMBERS

14. We shall give to addition in Algebra its arithmetical meaning, *so long as the numbers to be added are positive integers or positive fractions.*

We may then attach any meaning we please to addition involving other forms of numbers, provided the new meanings are not inconsistent with principles previously established.

15. In adding a positive number and a negative, or two negative numbers, our methods must be in accordance with the principles of § 10.

If a man has assets of \$5, and then incurs liabilities of \$3, he will be worth \$2.

If he has assets of \$3, and then incurs liabilities of \$5, he will be in debt to the amount of \$2.

If he has liabilities of \$5, and then incurs liabilities of \$3, he will be in debt to the amount of \$8.

Now with the notation of § 11, incurring liabilities of \$3 may be regarded as adding $-\$3$ to his property.

Whence, the sum of $+\$5$ and $-\$3$ is $+\$2$;
the sum of $-\$5$ and $+\$3$ is $-\$2$;
and the sum of $-\$5$ and $-\$3$ is $-\$8$.

Or, omitting reference tc the *unit*,

$$(+5)+(-3)=+2;$$
$$(-5)+(+3)=-2;$$
$$(-5)+(-3)=-8.$$

To indicate the addition of $+5$ and -3, they must be enclosed in *parentheses* (§ 7).

We then have the following rules:

To add a positive and a negative number, subtract the less absolute value (§ 13) from the greater, and prefix to the result the sign of the number having the greater absolute value.

To add two negative numbers, add their absolute values, and prefix a negative sign to the result.

16. Examples.

1. Find the sum of $+10$ and -3.

Subtracting 3 from 10, the result is 7.
Whence, $(+10)+(-3)=+7.$

2. Find the sum of -12 and $+6$.

Subtracting 6 from 12, the result is 6.
Whence, $(-12)+(+6)=-6.$

3. Add -9 and -5.

The sum of 9 and 5 is 14.
Whence, $(-9)+(-5)=-14.$

EXERCISE 5

Find the values of the following:

1. $(-6)+(-7)$.
2. $(+8)+(-3)$.
3. $(-9)+(+5)$.
4. $(+4)+(-11)$.
5. $(-13)+(-18)$.
6. $(-42)+(+57)$.
7. $(-34)+(+82)$.
8. $\left(-\frac{5}{9}\right)+\left(-\frac{1}{6}\right)$
9. $\left(+\frac{9}{8}\right)+\left(-\frac{8}{7}\right)$.
10. $(-15\frac{1}{3})+(+12\frac{7}{9})$.
11. $(+17\frac{7}{8})+(-10\frac{5}{12})$.
12. $(-14\frac{5}{6})+(-21\frac{3}{14})$.

MULTIPLICATION OF POSITIVE AND NEGATIVE NUMBERS

17. If one algebraic expression is multiplied by another, the first is called the **Multiplicand**, and the second the **Multiplier.**

18. We shall retain for multiplication, in Algebra, its arithmetical meaning, *so long as the multiplier is a positive integer or a positive fraction.* That is, to multiply a number by a positive integer is to *add* the multiplicand as many times as there are units in the multiplier.

For example, to multiply -4 by 3, we add -4 three times.

Thus, $(-4)\times(+3)=(-4)+(-4)+(-4)=-12.$

19. In Arithmetic, the product of two numbers is the same in whichever *order* they are multiplied, that is, whichever is taken as the multiplier.

Thus, 3×4 and 4×3 are each equal to 12.

If we could assume this law to hold for the product of a positive number by a negative, we should have

$$(+3)\times(-4)=(-4)\times(+3)=-12\ (\S\ 18)=-(3\times4).$$

Then, if the above law is to hold, we must give the following meaning to multiplication by a negative number:

To multiply a number by a negative number is to multiply it by the absolute value (§ 13) of the multiplier, and change the sign of the result.

Thus, to multiply $+4$ by -3, we multiply $+4$ by $+3$, giving $+12$, and change the sign of the result.

That is, $(+4)\times(-3)=-12.$

Again, to multiply -4 by -3, we multiply -4 by $+3$, giving -12 (§ 18), and change the sign of the result.

That is, $(-4)\times(-3)=+12.$

20. From §§ 18 and 19 we derive the following rule:

To multiply one number by another, multiply together their absolute values.

Make the product *plus* when the multiplicand and multiplier are of *like* sign, and *minus* when they are of *unlike* sign.

21. Examples.

1. Multiply $+8$ by -5.

By the rule, $(+8)\times(-5) = -(8\times 5) = -40$.

2. Multiply -7 by -9.

By the rule, $(-7)\times(-9) = +(7\times 9) = +63$.

3. Find the numerical value when $a=4$ and $b=-7$, of $(a+b)^3$.

We have, $(a+b)^3 = (4-7)(4-7)(4-7)$
$= (-3)(-3)(-3) = -27$.

EXERCISE 6

Find the values of the following:

1. $(+5)\times(-4)$.
2. $(-11)\times(+3)$.
3. $(-8)\times(-7)$.
4. $(+9)\times(-6)$.
5. $(-12)\times(+9)$.
6. $(-24)\times(-5)$.
7. $(-14)\times(+15)$.
8. $(+27)\times(-19)$.
9. $\left(-\frac{7}{8}\right)\times\left(-\frac{9}{5}\right)$.

III. ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS. PARENTHESES

22. A **Monomial**, or **Term**, is an expression (§ 8) whose parts are not separated by the sign $+$ or $-$; as $2x^2$, $-3ab$, or 5.

$2x^2$, $-3ab$, and $+5$ are called the *terms* of the expression $2x^2-3ab+5$.

A **Positive Term** is one preceded by a $+$ sign; as $+5a$. If no sign is expressed, the term is understood to be positive.

A **Negative Term** is one preceded by a $-$ sign; as $-3ab$. The $-$ sign must never be omitted before a negative term.

23. If two or more numbers are multiplied together, each of them, or the product of any number of them, is called a **Factor** of the product.

Thus, a, b, c, ab, ac, and bc are factors of the product abc.

24. Any factor of a product is called the **Coefficient** of the product of the remaining factors.

Thus, in $2ab$, 2 is the coefficient of ab, $2a$ of b, a of $2b$, etc.

25. If one factor of a product is expressed in *Arabic numerals*, and the other in *letters*, the former is called the *numerical coefficient* of the latter.

Thus, in $2ab$, 2 is the numerical coefficient of ab.

If no numerical coefficient is expressed, the coefficient 1 is understood; thus, a is the same as $1a$.

26. By § 20, $(-3)\times a = -(3\times a) = -3a$.

That is, $-3a$ is the product of -3 and a.

Then, -3 is the *numerical coefficient* of a in $-3a$.

Thus, *in a negative term as in a positive, the numerical coefficient includes the sign.*

27. Similar or Like Terms are those which either do not differ at all, or differ only in their numerical coefficients; as $2x^2y$ and $-7x^2y$.

Dissimilar or Unlike Terms are those which are not similar; as $3x^2y$ and $3xy^2$.

ADDITION OF MONOMIALS

28. The result of addition is called the Sum.

29. The adding of b to a is expressed $a+b$. The sum is $(a+b)$. But where no ambiguity is to be feared, parentheses may be omitted.

30. The addition of monomials is effected by uniting them with their respective signs.

Thus, the sum of a, $-b$, c, $-d$, and $-e$ is

$$a-b+c-d-e.$$

31. We assume that the terms can be united *in any order*, provided each has its proper sign.

Hence, the result of § 30 can also be expressed

$$c+a-e-d-b,\ -d-b+c-e+a, \text{ etc.}$$

32. To multiply $5+3$ by 4, we multiply 5 by 4, and then 3 by 4, and add the second result to the first.

Thus, $(5+3)4=5\times4+3\times4$.

We then assume that to multiply $a+b$ by c, we multiply a by c, and b by c, and add the second result to the first.

Thus, $(a+b)c=ac+bc$.

33. Addition of Similar Terms (§ 27).

1. Required the sum of $5a$ and $3a$.

We have, $5a+3a=(5+3)a$ (§ 32)

$=8a$.

That is, we do not add the a's but the coefficients of the a's.

2. Required the sum of $-5a$ and $-3a$.

We have, $(-5a)+(-3a)=(-5)\times a+(-3)\times a$ (§ 26)

$=[(-5)+(-3)]\times a$ (§ 32)

$=(-8)\times a$ (§ 15)

$=-8a$. (§ 26)

3. Required the sum of $5a$ and $-3a$.

We have, $5a+(-3)a=[5+(-3)]\times a$ (§ 32)

$=2a$. (§ 15)

4. Required the sum of $-5a$ and $3a$.

We have, $(-5)a+3a=[(-5)+3]\times a$ (§ 32)

$=(-2)\times a$ (§ 15) $=-2a$.

Therefore, to add two similar terms, find the sum of their numerical coefficients (§§ 15, 25, 26), and affix to the result the common letters.

5. Find the sum of $2a$, $-a$, $3a$, $-12a$, and $6a$.

Since the additions may be performed in any order, we may add the positive terms first, and then the negative terms, and finally combine these two results.

The sum of $2a$, $3a$, and $6a$ is $11a$.

The sum of $-a$ and $-12a$ is $-13a$.

Hence, the required sum is $11a+(-13a)$, or $-2a$.

6. Add $3(a-b)$, $-2(a-b)$, $6(a-b)$, and $-4(a-b)$.

The sum of $3(a-b)$ and $6(a-b)$ is $9(a-b)$.
The sum of $-2(a-b)$ and $-4(a-b)$ is $-6(a-b)$.
Then, the result is $[9+(-6)](a-b)$, or $3(a-b)$.

If the terms are not all similar, we may combine the similar terms, and unite the others with their respective signs (§ 30).

7. Required the sum of $12\,a$, $-5\,x$, $-3\,y^2$, $-5\,a$, $8\,x$, and $-3\,x$.

The sum of $12\,a$ and $-5\,a$ is $7\,a$.
The sum of $-5\,x$, $8\,x$, and $-3\,x$ is 0.
Then, the required sum is $7\,a-3\,y^2$.

EXERCISE 7

Add the following:

1. $8\,m$ and $4\,m$.
2. $12\,a$ and $-5\,a$.
3. $12\,a$ and $-16\,a$.
4. $-12\,a$ and $-5\,a$.
5. $-8\,y^2$ and $-20\,y^2$.
6. $-15\,cd$ and $13\,cd$.
7. $24\,a^2b$ and $-23\,a^2b$.
8. $31\,c^5d^2$ and $-31\,c^5d^2$.
9. $-6(c+d)$ and $-4(c+d)$.
10. $-5(x^2+y^2)$ and $-9(x^2+y^2)$.
11. $7\,x$, $4\,x$, and $-3\,x$.
12. $16\,a$, $-5\,a$, $-3\,a$, and a.
13. $2\,a$, $-5\,a$, and $-11\,a$.
14. $3\,xyz$, $6\,xyz$, and $-9\,xyz$.
15. $8(x+y)$, $-14(x+y)$, and $3(x+y)$.
16. $8\,n^2$, $-n^2$, $14\,n^2$, $-4\,n^2$, and $7\,n^2$.
17. $3\,a^4b^2$, $-5\,a^4b^2$, a^4b^2, $-9\,a^4b^2$, and $20\,a^4b^2$.
18. $3\,ax$, $-4\,bx$, $5\,ax$, and $-2\,bx$.
19. $3(a+b)$, $4(a-b)$, $-2(a+b)$, and $6(a-b)$.
20. $4\,h$, $3\,k$, $-5\,a$, $2\,k$, $-h$, and $2\,a$.

ADDITION OF POLYNOMIALS

34. A **Polynomial** is an algebraic expression consisting of more than one term; as $a+b$, or $2\,x^2-xy-3\,y^3$.

A polynomial is also called a *multinomial*.

A **Binomial** is a polynomial of *two* terms; as $a+b$.

A **Trinomial** is a polynomial of *three* terms; as $a+b-c$.

35. A polynomial is said to be *arranged* according to the *descending* powers of any letter, when the term containing the highest power of that letter is placed first, that having the next lower immediately after, and so on.

Thus, $x^4+3x^3y-2x^2y^2+3xy^3-4y^4$

is arranged according to the descending powers of x.

The term $-4y^4$, which does not involve x at all, is regarded as containing the lowest power of x in the above expression.

A polynomial is said to be arranged according to the *ascending* powers of any letter, when the term containing the lowest power of that letter is placed first, that having the next higher immediately after, and so on.

Thus, $x^4+3x^3y-2x^2y^2+3xy^3-4y^4$

is arranged according to the ascending powers of y.

36. Addition of Polynomials.

Let it be required to add $b+c$ to a.

Since $b+c$ is the sum of b and c (§ 29), we may add $b+c$ to a by adding b and c separately to a.

Then, $$a+(b+c)=a+b+c.$$

(To indicate the addition of $b+c$, we write it in parenthesis.)

The above assumes that, to add the sum of a set of terms, we add the terms separately.

37. From § 36 we have the following rule:

To add a polynomial to a quantity, add its terms with their signs unchanged.

1. Add $6a-7x^2$, $3x^2-2a+3y^3$, and $2x^2-a-mn$.

We set the expressions down one underneath the other, similar terms being in the same vertical column.

We then find the sum of the terms in each column, and write the results with their respective signs; thus,

$$\begin{array}{r} 6a-7x^2 \\ -2a+3x^2+3y^3 \\ -\ a+2x^2 \qquad -mn \\ \hline 3a-2x^2+3y^3-mn \end{array}$$

2. Add $4x-3x^2-11+5x^3$, $12x^2-7-8x^3-15x$, and $14+6x^3+10x-9x^2$.

It is convenient to arrange each expression in *descending* powers of x (§ 35); thus,

$$\begin{array}{r} 5x^3-\ 3x^2+\ 4x-11 \\ -8x^3+12x^2-15x-\ 7 \\ \hline 6x^3-\ 9x^2+10x+14 \\ \hline 3x^3\qquad -\ x-\ 4 \end{array}$$

3. Add $9(a+b)-8(b+c)$, $-3(b+c)-7(c+a)$, and $4(c+a)-5(a+b)$.

$$\begin{array}{r} 9(a+b)-\ 8(b+c) \\ -\ 3(b+c)-7(c+a) \\ -5(a+b)\qquad +4(c+a) \\ \hline 4(a+b)-11(b+c)-3(c+a) \end{array}$$

4. Add $\frac{3}{4}a+\frac{2}{5}b-\frac{1}{3}c$ and $\frac{1}{6}a-\frac{4}{5}b+\frac{5}{7}c$.

$$\begin{array}{r} \frac{3}{4}a+\ \frac{2}{5}b-\ \frac{1}{3}c \\ \frac{1}{6}a-\ \frac{4}{5}b+\ \frac{5}{7}c \\ \hline \frac{11}{12}a-\frac{14}{15}b+\frac{8}{21}c \end{array}$$

EXERCISE 8

Add the following: (Results may be checked as in Chap. XVII.)

1.	2.	3.
$2a-5b$	$-\ 4x^2+3y^2$	$-8xy+2st$
$-7a+6b$	x^2-4y^2	$+2xy-7st$
$9a-\ b$	$-11x^2+8y^2$	$-3xy+5st$

4. $7d-4r-6n$ and $3d+9r+2n$.
5. $5a^2-4ab+b^2$, $4a^2+4ab+5b^2$, and $-9a^2+6b^2$.
6. $2m-3x+f$, $m+x-f$, and $m+f$.
7. $3bu+2pk-qt$, $5bu-7pk+2qt$.
8. $b-2+3b^2-8b^3$, $b+6-b^2+7b^3$, and $b+2b^2-4b^3$.
9. $3(a+b)-7(b+c)$, $5(a+b)+5(b+c)$, and $-2(a+b)-3(b+c)$.
10. $\frac{1}{2}a-\frac{1}{3}b+\frac{2}{5}c$ and $-\frac{3}{4}a+\frac{4}{5}b-\frac{1}{10}c$.
11. $4t+3u-5c$, $-2t-a+3c$, $2a-9c+2u$, and $5t+3a-4u$.
12. $\frac{8}{15}x+\frac{2}{9}y+\frac{13}{10}z$ and $\frac{3}{10}x-\frac{7}{9}y-\frac{1}{2}z$.

13. Add these equations (Ax. 5, § 4), then find the value of x:

$$\begin{cases} x+y=5, \\ x-y=7. \end{cases}$$

After x is known can you find y?

14. Find y by adding these equations:

$$\begin{cases} 5x+2y=16, \\ -5x+3y=-1. \end{cases}$$

What value has x?

15. Find x and y in these equations:

$$\begin{cases} 2x+3y=11, \\ x-3y=1. \end{cases}$$

Add the following:

16. $14(x+y)-17(y+z)$, $4(y+z)-9(z+x)$, and $-3(x+y)-7(z+x)$.

17. $6c+2a-3b$, $4d-7c+12a$, $8b-5d+c$, and $-10a-11b+9d$.

18. $-7(a-b)^2+8(a-b)+2$, $4(a-b)^2-5(a-b)$, and $3(a-b)^2-9$.

EXERCISE 9

	1st No.		2nd No.		Sum.		1st No.		2nd No.		Sum.
1.	$+8$	$+$	?	$=$	5	6.	x	$+$	?	$=$	$x+y$
2.	-10	$+$	?	$=$	-7	7.	a	$+$	?	$=$	$a-b$
3.	$+10$	$+$	?	$=$	-7	8.	a	$+$	?	$=$	b
4.	$+6$	$+$	?	$=$	11	9.	$-b$	$+$	?	$=$	a
5.	-3	$+$	?	$=$	-9	10.	c	$+$	?	$=$	b

38. In the above examples we have given the sum of two numbers and one of the numbers to find the other number.

SUBTRACTION OF MONOMIALS

39. Subtraction is the process of finding one of two numbers when their sum and one of them is given.

The Minuend is the sum of the numbers.

The Subtrahend is the given number.

The Difference is the required number.

40. Therefore, to subtract one number from another is to find a number, which added to the subtrahend will produce the minuend.

For example, to subtract 3 from 10, we find the number, which, added to 3, will produce 10. By remembering the result in addition such number is seen to be 7. Thus 7 is our *difference*.

To subtract -4 from 9, find the number which, added to -4, will produce 9. By inspection this number is evidently 13.

Subtract -6 from -8.

-6 plus -2 gives -8,

hence our *difference* is -2.

Subtract $+3$ from -9.

$$3+(-12)=-9,$$

hence -12 is our *difference*.

EXERCISE 10

Subtract the following:

1. 7 from 2.
2. 3 from -8.
3. -11 from -10.
4. -3 from 8.
5. -6 from -11.
6. 36 from 12.
7. 6 from 13.
8. -9 from 3.

9.	10.	11.	12.
$9x$	$4a$	$-4a$	$13t$
$3x$	$-5a$	$-7a$	$-t$

41. Notice that in each of the above examples the result is the same as if we had changed the sign of the subtrahend and proceeded as if adding the subtrahend to the minuend.

42. Similarly, from § 41 this rule follows:

To subtract one number from another, change the sign of the subtrahend and proceed as in addition. (The sign of the subtrahend must be changed *mentally*.)

EXERCISE 11

Subtract the following:

(The accuracy of all results may be checked by adding the difference to the subtrahend.)

1. $5ax$ from ax.
2. $3abc$ from $-9abc$.
3. $14a^2b^2$ from $11a^2b^2$.
4. $15(a-b)$ from $19(a-b)$.

5. $\frac{1}{3}my$ from $\frac{1}{2}my$.
6. $-11\,c^2s$ from $-6\,c^2s$.
7. $-21\,cy$ from $13\,cy$.
8. From $8\,a$ take $3\,b$.
9. From $7\,x$ take $-2\,y$.
10. From $-3\,a$ take $4\,b^2$.

SUBTRACTION OF POLYNOMIALS

43. Since a polynomial may be regarded as the sum of its separate terms (§ 30), we have the following rule:

To subtract a polynomial, change the sign of each of its terms, and add the result to the minuend.

1. Subtract $7\,ab^2-9\,a^2b+8\,b^3$ from $5\,a^3-2\,a^2b+4\,ab^2$.

It is convenient to place the subtrahend under the minuend, so that similar terms shall be in the same vertical column.

We then *mentally* change the sign of each term of the subtrahend, and add the result to the minuend; thus,

$$\begin{array}{r} 5\,a^3-2\,a^2b+4\,ab^2 \\ -9\,a^2b+7\,ab^2+8\,b^3 \\ \hline 5\,a^3+7\,a^2b-3\,ab^2-8\,b^3 \end{array}$$

2. Subtract the sum of $9\,x^2-8\,x+x^3$ and $5-x^2+x$ from $6\,x^3-7\,x-4$.

We change the sign of each expression which is to be subtracted, and add the results.

$$\begin{array}{rrr} 6\,x^3 & & -7\,x-4 \\ -\,x^3 & -9\,x^2 & +8\,x \\ & +\,x^2 & -\,x-5 \\ \hline 5\,x^3 & -8\,x^2 & -9 \end{array}$$

EXERCISE 12

Subtract the following:

1. $x^2+13\,x-11$ from $-3\,x^2+6\,x-5$.
2. $-2\,m^2-4\,mn+9\,n^2$ from $8\,m^2-7\,mn+14\,n^2$.
3. $ab+bc+ca$ from $ab-bc+ca$.
4. From $9\,a+4\,h-5\,k$ take $9\,a-4\,h+5\,k$.
5. From $6\,x^3-5\,x^2+4\,x-3$ take $x^3-3\,x^2-2\,x+1$.
6. From $11\,a-9\,b+2\,z$ subtract $-3\,z+2\,a-14\,b$.

7. Take $-8(h+k)+3(h-k)$ from $(h+k)-4(h-k)$.

8. Subtract $74\,z^2-47\,zk+30\,k^2$ from $24\,k^2-30\,zk+10\,z^2$.

9. From $7\,v-8\,s+75\,t^2$ take $-16\,v+19\,s$.

10. What must be added to $4\,g+18\,z^2-x$ to give 0?

11. By how much does $8\,x^3-7\,x^2+5\,x-1$ exceed $x^3+14\,x^2-3\,x+7$?

12. From $x^3-11\,x+4$ subtract $8\,x^2-3\,x-1$.

13. From $a^2+2\,ab+b^2$ take $a^2-2\,ab+b^2$.

14. Find the sum of $a^2+2\,ab+b^2$ and $a^2-2\,ab+b^2$.

15. From the sum of $x^3+4\,x^2+4\,x$ and $2\,x^2+8\,x+8$ take $6\,x^2+12\,x$.

16. From the sum of $a^3-2\,a^2b+ab^2$ and $-a^2b+2\,ab^2-b^3$ take the sum of $a^3+2\,a^2b+ab^2$ and $a^2b+2\,ab^2+b^3$.

17. From $\frac{4}{5}\,s-\frac{5}{6}\,a+\frac{2}{3}\,b$ take $\frac{1}{5}\,s+\frac{1}{2}\,a-\frac{1}{6}\,b$.

18. From $\frac{3}{2}\,gt^2+v-\frac{1}{3}\,t$ take $gt^2-\frac{4}{5}\,v+6\,t$.

19. Take $a^4-6\,a^3-15\,a^2-8\,a+4$ from $7\,a^4+3\,a^3-5\,a^2-11\,a-9$.

20. From $\frac{1}{2}\,m-\frac{1}{3}\,n+\frac{2}{7}\,p$ take $\frac{3}{5}\,m-\frac{3}{4}\,n+\frac{7}{8}\,p$.

21. From $n^4-10\,n^3x-n^2x^2+8\,nx^3+3\,x^4$ take $5\,n^4+4\,n^3x-9\,n^2x^2+2\,nx^3-12\,x^4$.

22. Take $18\,x^4-8\,x+6\,x^5+12-8\,x^3$ from $-10\,x^3+2-15\,x^2+11\,x^5-4\,x$.

23. Take $a^5-10\,a^3b^2+13\,a^2b^3-7\,ab^4-5\,b^5$ from $9\,a^5+3\,a^4b+6\,a^3b^2-a^2b^3-16\,b^5$.

24. From the sum of $2\,x^2-5\,xy-y^2$ and $7\,x^2-3\,xy+9\,y^2$ subtract $4\,x^2-6\,xy+8\,y^2$.

25. From 0 subtract the sum of $4\,a^2$ and $3\,a-5\,a^2-1$.

Add the following pairs of equations to find x, subtract them (Ax. 6, § 4) to find y. Verify results by substituting the values of x and y in the given equations:

26. $\begin{cases} x+y=5, \\ x-y=1. \end{cases}$ 27. $\begin{cases} 2\,x+5\,y=16, \\ 2\,x-5\,y=-4. \end{cases}$ 28. $\begin{cases} 5\,y+x=9, \\ 5\,y-x=1. \end{cases}$

PARENTHESES

44. Removal of Parentheses.

By § 30, $a+(b-c)=a+b-c$. **Hence,**

Parentheses preceded by a + sign may be removed without changing the signs of the terms enclosed.

Again, by § 43, $a-(b-c)=a-b+c$. **Hence,**

Parentheses preceded by a − sign may be removed if the sign of each term enclosed be changed.

The above rules apply equally to the removal of the *brackets*, *braces*, or *vinculum* (§ 7).

It should be noticed in the case of the latter that the sign apparently prefixed to the first term underneath is in reality prefixed to the vinculum; thus, $+\overline{a-b}$ means the same as $+(a-b)$, and $-\overline{a-b}$ the same as $-(a-b)$.

45. 1. Remove the parentheses from

$$2a-3b-(5a-4b)+(4a-b).$$

By the rules of § 44, the expression becomes

$$2a-3b-5a+4b+4a-b=a.$$

Parentheses sometimes enclose others; in this case they may be removed in succession by the rules of § 44.

Beginners should remove one at a time, commencing with the *innermost* pair; after a little practice, they should be able to remove several signs of aggregation at one operation, in which case they should commence with the outermost pair.

2. Simplify $4x-\{3x+(-2x-\overline{x-a})\}$.

We remove the vinculum first, then the parentheses, and finally the braces.

Thus,
$$\begin{aligned}&4x-\{3x+(-2x-\overline{x-a})\}\\&=4x-\{3x+(-2x-x+a)\}\\&=4x-\{3x-2x-x+a\}\\&=4x-3x+2x+x-a=4x-a.\end{aligned}$$

EXERCISE 13

1. What is the sign of $2x$ in $3x^3-4c-(2x+1)$?

2. What is the sign of a in $4a^3-\overline{a-4c^2}+9x^2$? What is the coefficient of a after the vinculum is removed?

Simplify by removing the signs of aggregation and then uniting similar terms:

3. $11\,a-(-6\,m+5\,c)-(3\,a+4\,c)$.
4. $4\,x-3\,y-[7\,y-d]+\{-4\,x-3\,y\}$.
5. $x^2+[-3\,x^2-(2\,y^2-2\,x^2)+2\,y^2]$.
6. $7\,t+u-\{6\,t-\overline{u+7}-3\}$.
7. $(a^2+2\,ab+b^2)-(a^2-2\,ab+b^2)$.

Compare Ex. 13, Exercise 12.

8. $x^3-(-3\,x^2y-3\,xy^2)+y^3-(x^3-[3\,x^2y-3\,xy^2+y^3])$.

Compare Ex. 16, Exercise 12.

9. $7\,x-\{-8\,y-\overline{10\,x-11\,y}\}$.
10. $a^3-(-6\,a^2-\overline{12\,a+8})-(a^3+12\,a)$.

Compare Ex. 15, Exercise 12.

46. Insertion of Parentheses. — To write terms in parenthesis, we take the converse of the rules of § 44.

Any number of terms may be written in parenthesis preceded by a + sign, without changing their signs.

Any number of terms may be written in parenthesis preceded by a − sign, if the sign of each term be changed.

Ex. Write the last three terms of $a-b+c-d+e$ in parenthesis preceded by a − sign.

Result, $a-b-(-c+d-e)$.

EXERCISE 14

In each of the following expressions, write the last three terms in parenthesis preceded by a − sign:

1. $a+b+c-d$.
2. $m^2+3\,m-2+h$.
3. $x^3-3\,x^2+3\,x-1$.
4. $4\,a^4-3\,a^3-2\,a^2-a$.
5. $8\,x^2-y^2-y+z$.
6. $a^2-b^2+c^2-d^2$.
7. $x^2-2\,xy-y^2-2\,yz-z^2$.
8. $2\,a^5-10\,a^4-8\,a^3+5\,a^2-6\,a+9$.
9. In each of the above results, write the last two terms in parenthesis in brackets preceded by a − sign.

47. Addition and Subtraction of Terms having Literal Coefficients. — To add two or more terms involving the same power of a certain letter, with literal, or numerical and literal, coefficients, it is convenient to put the coefficient of this letter in parenthesis.

1. Add ax and $2x$.

By § 32, $\quad ax+2x=(a+2)x.$

2. Add $(2m+n)y$ and $(m-3n)y$.

$$(2m+n)y+(m-3n)y=[(2m+n)+(m-3n)]y$$
$$=(2m+n+m-3n)y=(3m-2n)y.$$

(The pupil should endeavor to put down the result in one operation.)

3. Subtract $(b-c)x^2$ from ax^2.

By §§ 32, 42, 44, $\quad ax^2-(b-c)x^2=[a-(b-c)]x^2$
$$=(a-b+c)x^2.$$

EXERCISE 15

Add the following:

1. ax and bx.
2. $3ab^2$ and $-4b^2$.
3. $-a^2b$ and $5a^2y$.
4. $3a^2bc$ and $-8cd$.
5. ab, bc and $-m^2b$.
6. c^2x^3 and $(a-3d)x^3$.
7. $(7a+4b)x^2$ and $(3m+n)x^2$.
8. $(4x-y)z^4$ and $(3x+c)z^4$.

Subtract the following:

9. $3cx$ from dx.
10. $-4my$ from $3cy$.
11. $-cxy$ from $-dxy$.
12. $(c+d)x$ from ax.
13. $(3c-4d)x^2$ from $(6c+9d)x^2$.

QUERIES

1. What is the difference between $4x$ and $3y$?
2. Express the sum of three times a certain number and four times the same number.
3. Do you make any distinction between a *factor* and a *coefficient*?
4. Regarding $3mxy$ and $5cdx$ as the coefficients of the expressions $3amxy$ and $5acdx$, find the sum. Regard ax as the common part and find the sum.
5. Does the sum of a and b, $(a+b)$, have the same significance to you as the sum of 5 and 2, $(5+2)$?

6. Express algebraically: the sum of 6 times the sum of a and b, and 9 times the sum of the same two numbers.

7. What must you add to a polynomial to produce 0? Give an example.

8. Did you work problem 7 by addition or subtraction? Is there any difference between the two processes as here used?

9. Have you noticed which forms of the signs of aggregation are used most? Does the vinculum appear often?

10. Subtract $3\,m(a+b)$ from $6\,m(a-b)$.

11 If your difference is x^3-4x+7 and your subtrahend is $3x^3+4x-9$, what is your minuend?

IV. MULTIPLICATION OF ALGEBRAIC EXPRESSIONS

48. The Rule of Signs.

If a and b are any two positive numbers, we have by § 20,

$$(+a)\times(+b)=+ab, \qquad (+a)\times(-b)=-ab,$$
$$(-a)\times(+b)=-ab, \qquad (-a)\times(-b)=+ab.$$

From these results we may state what is called the **Rule of Signs** in multiplication, as follows:

The product of two terms of like sign is positive; the product of two terms of unlike sign is negative.

49. We have by § 48,

$$(-a)\times(-b)\times(-c)=(ab)\times(-c)$$
$$=-abc; \qquad (1)$$
$$(-a)\times(-b)\times(-c)\times(-d)=(-abc)\times(-d), \text{ by (1)},$$
$$=abcd; \text{ etc.}$$

That is, the product of three negative terms is negative; the product of four negative terms is positive; and so on.

In general, the product of any number of terms is positive or negative according as the number of negative terms is even or odd.

50. The Law of Exponents.

Let it be required to multiply a^3 by a^2.

By § 6, $a^3 = a \times a \times a$,

and $a^2 = a \times a$.

Whence, $a^3 \times a^2 = a \times a \times a \times a \times a = a^5$.

The general case. — Let it be required to multiply a^m by a^n, where m and n are any positive integers.

We have $a^m = a \times a \times \cdots$ to m factors,

and $a^n = a \times a \times \cdots$ to n factors.

Then, $a^m \times a^n = a \times a \times \cdots$ to $m+n$ factors $= a^{m+n}$.

(The *Sign of Continuation*, $\cdots$, is read "*and so on.*")

Hence, the exponent of a letter in the product is equal to its exponent in the multiplicand plus its exponent in the multiplier.

This is called the *Law of Exponents for Multiplication.*

A similar result holds for the product of three or more powers of the same letter.

Thus, $a^3 \times a^4 \times a^5 = a^{3+4+5} = a^{12}$.

MULTIPLICATION OF MONOMIALS

51. 1. Let it be required to multiply $7a$ by $-2b$.

By § 26, $-2b = (-2) \times b$.

Then, $7a \times (-2b) = 7a \times (-2) \times b$

$= 7 \times (-2) \times a \times b = -14ab$. (§ 48)

In the above solution, we assume that *the factors of a product can be written in any order.*

2. Required the product of $-2a^2b^3$, $6ab^5$, and $-7a^4c$.

$(-2a^2b^3) \times 6ab^5 \times (-7a^4c)$

$= (-2)a^2b^3 \times 6ab^5 \times (-7)a^4c$

$= (-2) \times 6 \times (-7) \times a^2 \times a \times a^4 \times b^3 \times b^5 \times c$

$= 84a^7b^8c$, by §§ 49 and 50.

We then have the following rule for the product of any number of monomials:

To the product of the numerical coefficients (§§ 25, 26, 49, 50) annex the letters; giving to each an exponent equal to the sum of its exponents in the factors.

3. Multiply $-5\,a^3b$ by $-8\,ab^3$.

$$(-5\,a^3b)\times(-8\,ab^3)=40\,a^{3+1}b^{1+3}=40\,a^4b^4.$$

4. Find the product of $4\,n^2$, $-3\,n^5$, and $2\,n^4$.

$$4\,n^2\times(-3\,n^5)\times 2\,n^4=-24\,n^{2+5+4}=-24\,n^{11}.$$

5. Multiply $-x^m$ by $7\,x^6$.

$$(-x^m)\times 7\,x^6=-7\,x^{m+6}.$$

6. Multiply $6(m+n)^4$ by $7(m+n)^3$.

$$6\,(m+n)^4\times 7\,(m+n)^3=42\,(m+n)^7.$$

EXERCISE 16

Multiply the following:

1. $4\,a^2$ by $7\,a^5$.
2. $6\,m^2d$ by $9\,md^2$.
3. $11\,ax$ by $-8\,ab$.
4. $-7\,cx$ by $-10\,rs$.
5. $-12\,x^3y$ by $9\,xy^3$.
6. $4\,x^ay^2z$ by $-11\,x^2y^mz^3$.
7. $3\,(x+y)^2$ by $16\,(x+y)^3$.
8. $-4\,(a-b)$ by $6\,(a-b)^2$.

Find the product of:

9. $(x-y)^2$, $4(x-y)^3$, and $-7\,(x-y)$.
10. $3\,m^2nz^3$, $-5\,mn^3z$, and $-6\,mnz^4$.
11. a^4c^2, $-2\,a^3c^4$, $-5\,ac^5$, and $6\,a^5bc$.
12. a^zb^y, $4\,a^2b$, and $-11\,a^{2z}b^{3y}$.

MULTIPLICATION OF POLYNOMIALS BY MONOMIALS

52. In § 32, we assumed that the product of $a+b$ by c was $ac+bc$. We then have the following rule for the product of a polynomial by a monomial:

Multiply each term of the multiplicand by the multiplier, and add the partial products.

Ex. Multiply $2\,x^2-5\,x+7$ by $-8\,x^3$.

$$\begin{aligned}&(2\,x^3-5\,x+7)\times(-8\,x^3)\\&\quad=(2\,x^3)\times(-8\,x^3)+(-5\,x)\times(-8\,x^3)+(7)\times(-8\,x^3)\\&\quad=-16\,x^5+40\,x^4-56\,x^3.\end{aligned}$$

The written work should stand as follows:

$$\begin{array}{r} 2\,x^3-\ 5\,x\ +\ 7\\ -\ 8\,x^3\\ \hline -16\,x^5+40\,x^4-56\,x^3\end{array}$$

EXERCISE 17

Multiply the following:

1. $12a-3$ by $5a$.
2. $3a^2b+7ab^2$ by $-9a^2b^2$.
3. $x^4-8x^3y^2+8xy^3$ by $-x^2y^2$.
4. $4m^4-3n^2-4$ by $7m^3$.
5. $5z^2$ by $8z^4-3z^2+13$.
6. $-9cd^3$ by $5c^3-10c^2d+2d^3$.
7. $8a^5-4a^6+9a^7$ by $-8a^4$.
8. $8x^n-3x^{2n}$ by $15x^{3n}$.
9. $-15a^2b+7b^3-4a^4$ by $-8a^5b^2$.
10. $4x^2y^2$ by $x^3+6x^2y-6xy^2+8y^3$.
11. $12a^2b-6ab^2-8b^3+a^3$ by $-8ab$.
12. $6a^3-8a^2c+7ac^2-11c^3$ by $14a^4c^3$.
13. $a^3-9a^2b+27ab^2-27b^3$ by $-3b$.
14. $\frac{1}{4}a^2-\frac{1}{3}ab+\frac{1}{9}b^2$ by $-\frac{1}{3}b$.
15. $\frac{8}{15}c^2-\frac{4}{5}b^2c-\frac{8}{10}c^3d$ by $5cd$.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS

53. Let it be required to multiply $a+b$ by $c+d$.

As in § 32, we multiply $a+b$ by c, and then $a+b$ by d, and add the second result to the first; that is,

$$(a+b)(c+d)=(a+b)c+(a+b)d$$
$$=ac+bc+ad+bd.$$

RULE: — **Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.**

54. 1. Multiply $3a-4b$ by $2a-5b$.

In accordance with the rule, we multiply $3a-4b$ by $2a$, and then by $-5b$, and add the partial products.

A convenient arrangement of the work is shown below, similar terms of the partial products being in the same vertical column.

$$\begin{array}{r} 3a-4b \\ 2a-5b \\ \hline 6a^2-8ab \\ \quad -15ab+20b^2 \\ \hline 6a^2-23ab+20b^2 \end{array}$$

The work may be checked by solving the example with the multiplicand and multiplier interchanged.

2. Multiply $4ax^2+a^3-8x^3-2a^2x$ by $2x+a$.

It is convenient to arrange the multiplicand and multiplier in the same order of powers of some common letter (§ 35), and write the partial products in the same order.

Arranging the expressions according to the descending powers of a, we have

$$\begin{array}{r} a^3-2a^2x+4ax^2-8x^3 \\ a+2x \\ \hline a^4-2a^3x+4a^2x^2-8ax^3 \\ \quad 2a^3x-4a^2x^2+8ax^3-16x^4 \\ \hline a^4 \qquad\qquad\qquad -16x^4 \end{array}$$

EXERCISE 18

Multiply the following:

1. $3x-4$ by $8x+5$.
2. $4a+b$ by $4a+b$.
3. $7t-4u$ by $11t-3u$.
4. $6ab+b^2$ by $3ab-5b^2$.
5. a^2-a+1 by $a+1$.
6. k^2-k-12 by $k-7$.
7. $2x+3$ by $2x-3$.
8. $3x+7$ by $3x-7$.
9. $\frac{1}{3}c+\frac{1}{4}d$ by $\frac{1}{3}c-\frac{1}{4}d$.
10. $\frac{1}{3}c+\frac{1}{4}d$ by $\frac{1}{3}c+\frac{1}{4}d$.
11. $\frac{1}{3}c-\frac{1}{4}d$ by $\frac{1}{3}c-\frac{1}{4}d$.

(Note carefully in what way examples 9 to 11, and 12 to 14 differ.)

12. $x+3$ by $x+5$.
13. $x-3$ by $x+5$.
14. $x+3$ by $x-5$.
15. a^2+a+1 by a^2-a+1.
16. $6(m+n)^2-5(m+n)+1$ by $7(m+n)-2$.
17. $3(a-b)^4-2(a-b)$ by $4(a-b)^2+(a-b)$.
18. $7t^2+8t-1$ by $7t^2-8t+1$.
19. $6ab+a^2+9$ by $3a^2-4+2ab$.

20. $2h^2-10h+5$ by $h^2+5h-10$.

21. $t^3-t^2u+tu^2-u^3$ by $t+u$.

22. $h^2-3hk+9k^2$ by $h+3k$.

23. x^2+xy+y^2 by $x-y$.

Note the similarity in examples 21–23.

24. $3a^x+7b^y$ by $3a^x-7b^y$.

25. $5(x+y)+7(x-y)$ by $3(x+y)-2(x-y)$.

26. e^x+e^y by e^x-e^y.

27. $a^3+2a^2b+2ab^2+b^3$ by $a^2-2ab+b^2$.

28. $4a^{m+5}b^2-3a^4b^c$ by $a^{m+2}b-2ab^{c-1}$.

29. $5x^4-6x^3-4x^2+2x-3$ by $3x-2$.

30. $a^3-3a^2x+3ax^2-x^3$ by $a^3+3a^2x+3ax^2+x^3$.

55. If the product has more than one term involving the same power of a certain letter, with literal, or numerical and literal, coefficients, we may put the coefficient of this letter in parentheses, as in § 47.

Ex. Multiply $x^2-ax-bx+ab$ by $x-a$.

$$\begin{array}{r} x^2-ax-bx+ab \\ \underline{x-a\qquad\qquad\qquad\qquad} \\ x^3-ax^2-bx^2+abx \\ \quad -ax^2+a^2x+abx-a^2b \\ \hline x^3-(2a+b)x^2+(a^2+2ab)x-a^2b \end{array}$$

As in § 47, $-2ax^2-bx^2$ is equivalent to $-(2a+b)x^2$, and $a^2x+2abx$ to $(a^2+2ab)x$.

EXERCISE 19

Multiply the following:

1. $x^2+ax+bx+ab$ by $x+c$.
2. $x^2-mx+nx-mn$ by $x-p$.
3. $x^2-bx-cx+bc$ by $x-a$.
4. $x^2+ax-bx-3ab$ by $x+b$.
5. $x^2+ax+2bx+2ab$ by $x-c$.
6. $x^2+px-5qx-5pq$ by $x-r$.
7. $x^2-3ax-bx+3ab$ by $x+2a$.
8. $x^2-4mx+nx-4mn$ by $x+3n$.

9. $x^2+3ax-2bx+6ab$ by $x-4c$.

10. $(a-b)x-3ab$ by $2x-(a-b)$.

11. $x^{2n}-5ax^n+4bx^n-2ab$ by x^n+c.

12. $(2a-1)x^2+(a+2)x-(a+3)$ by $(a-2)x-a$.

56. *Ex.* Simplify $(a-2x)^2-2(3a+x)(a-x)$.

To simplify the expression, we first multiply $a-2x$ by itself (§ 6); we then find the product of 2, $3a+x$, and $a-x$, and subtract the second result from the first.

$$\begin{array}{r} a-2x \\ a-2x \\ \hline a^2-2ax \\ -2ax+4x^2 \\ \hline a^2-4ax+4x^2 \end{array} \qquad \begin{array}{r} 3a+x \\ a-x \\ \hline 3a^2+ax \\ -3ax-x^2 \\ \hline 3a^2-2ax-x^2 \\ 2 \\ \hline 6a^2-4ax-2x^2 \end{array}$$

Subtracting the second result from the first, we have

$$a^2-4ax+4x^2-6a^2+4ax+2x^2=-5a^2+6x^2.$$

EXERCISE 20

Simplify the following:

1. $(3a+5)(2a-8)+(4a-7)(a+6)$.
2. $(3x+2)(4x+3)-(3x-2)(4x-3)$.
3. $(a-2x)(b+3y)+(a+2x)(b-3y)$.
4. $(3m+1)^2(3m-1)^2$.
5. $(x-y)(x^2-y^2)-(x+y)(x^2+y^2)$.
6. $(2a+3b)^2-4(a-b)(a+5b)$.
7. $[3x-(5y+2z)][3x-(5y-2z)]$.
8. $[m+2n-(2m-n)][2m+n-(m-2n)]$.
9. $(a+b+c)^2-(a-b-c)^2$.
10. $(a+2)(a+3)(a-4)+(a-2)(a-3)(a+4)$.
11. $(\frac{5}{2}x-\frac{4}{3}y+\frac{1}{4}z)^2$.
12. $[2x^2+(3x-1)(4x+5)][5x^2-(4x+3)(x-2)]$.
13. $(a+2b-c-3d)^2$.

14. $(a-2)(a+3)-(a-3)(a+4)-(a-4)(a+5)$.

15. $(x+2)(2x-1)(3x-4)-(x-2)(2x+1)(3x+4)$.

DEFINITIONS

57. A monomial is said to be *rational and integral* when it is either a number expressed in Arabic numerals, or a single letter with unity for its exponent, or the product of two or more such numbers or letters.

Thus, $3a^2b^3$, being equivalent to $3 \cdot a \cdot a \cdot b \cdot b \cdot b$, is rational and integral.

A polynomial is said to be rational and integral when each term is rational and integral; as $2x^2-\frac{3}{4}ab+c^3$.

58. If a term has a literal portion which consists of a single letter with unity for its exponent, the term is said to be of the *first degree*. Thus, $2a$ is of the first degree.

The *degree* of any rational and integral monomial (§ 57) is the number of terms of the first degree which are multiplied together to form its literal portion.

Thus, $5ab$ is of the *second* degree; $3a^2b^3$, being equivalent to $3 \cdot a \cdot a \cdot b \cdot b \cdot b$, is of the *fifth* degree; etc.

The degree of a rational and integral monomial equals the sum of the exponents of the letters involved in it.

Thus, ab^4c^3 is of the *eighth* degree.

The degree of a rational and integral polynomial is the degree of its term of highest degree.

Thus, $2a^2b-3c+d^2$ is of the *third* degree.

Frequently the degree of a term or polynomial with respect to some particular letter is required. Thus $3a^2x^3-4bxy^6+2c^4$ is of the *third* degree in x.

59. Homogeneity. — *Homogeneous terms* are terms of the same degree. Thus, a^4, $3b^3c$, and $-5x^2y^2$ are homogeneous terms.

A polynomial is said to be homogeneous when its terms are homogeneous; as $a^3+3b^2c-4xyz$.

V. DIVISION OF ALGEBRAIC EXPRESSIONS

60. Division, in Algebra, is the process of finding one of two numbers, when their product and the other number are given.

The Dividend is the product of the numbers.

The Divisor is the given number.

The Quotient is the required number.

61. The Rule of Signs. — Since the dividend is the product of the divisor and quotient, the equations of § 48 may be written as follows:

$$\frac{+ab}{+a}=+b,\quad \frac{-ab}{-a}=+b,\quad \frac{-ab}{+a}=-b,\text{ and }\frac{+ab}{-a}=-b.$$

We may state the Rule of Signs in division as follows:

The quotient of two terms of like sign is positive; the quotient of two terms of unlike sign is negative.

62. Let
$$\frac{a}{b}=x. \tag{1}$$

Then, since the dividend is the product of the divisor and quotient, we have
$$a=bx.$$

Multiply each of these equals by c (Ax. 7, § 4),
$$ac=bcx.$$

Regarding ac as the dividend, bc as the divisor, and x as the quotient, this may be written
$$\frac{ac}{bc}=x. \tag{2}$$

From (1) and (2), $\frac{ac}{bc}=\frac{a}{b}$. (Ax. 4, § 4)

That is, a factor common to the dividend and divisor can be removed, or cancelled.

63. The Law of Exponents for Division. — Let it be required to divide a^5 by a^2.

By § 6,
$$\frac{a^5}{a^2}=\frac{a\times a\times a\times a\times a}{a\times a}.$$

Cancelling the common factor $a \times a$ (§ 62), we have

$$\frac{a^5}{a^2} = a \times a \times a = a^3.$$

Consider the general case:

Let it be required to divide a^m by a^n, where m and n are any positive integers such that m is greater than n.

We have, $$\frac{a^m}{a^n} = \frac{a \times a \times a \times \cdots \text{ to } m \text{ factors}}{a \times a \times a \times \cdots \text{ to } n \text{ factors}}.$$

Cancelling the common factor $a \times a \times a \times \cdots$ to n factors,

$$\frac{a^m}{a^n} = a \times a \times a \times \cdots \text{ to } m-n \text{ factors} = a^{m-n}.$$

Hence, the exponent of a letter in the quotient is equal to its exponent in the dividend, minus its exponent in the divisor.

This is called the *Law of Exponents for Division.*

DIVISION OF MONOMIALS

64. 1. Let it be required to divide $-14\,a^2b$ by $7\,a^2$.

By § 51, $$\frac{-14\,a^2b}{7\,a^2} = \frac{(-2) \times 7 \times a^2 \times b}{7 \times a^2}.$$

Cancelling the common factors 7 and a^2 (§ 62), we have

$$\frac{-14\,a^2b}{7\,a^2} = (-2) \times b = -2\,b.$$

Then to find the quotient of two monomials:

To the quotient of the numerical coefficients annex the letters, giving to each an exponent equal to its exponent in the dividend minus its exponent in the divisor, and omitting any letter having the same exponent in the dividend and divisor.

The work may be checked by multiplying the divisor by the quotient.

2. Divide $54\,a^5b^3c^2$ by $-9\,a^4b^3$.

$$\frac{54\,a^5b^3c^2}{-9\,a^4b^3} = -6\,a^{5-4}c^2 = -6\,ac^2.$$

3. Divide $-2\,x^{2m}y^nz^r$ by $-x^my^nz^5$.

$$\frac{-2\,x^{2m}y^nz^r}{-x^my^nz^5} = 2\,x^{2m-m}z^{r-5} = 2\,x^mz^{r-5}.$$

4. Divide $35(a-b)^7$ by $7(a-b)^4$.

$$\frac{35(a-b)^7}{7(a-b)^4} = 5(a-b)^3.$$

EXERCISE 21

Divide the following:

1. 30 by -5.
2. -42 by 6.
3. -48 by -4.
4. -64 by 8.
5. -135 by -9.
6. 176 by -11.
7. $-\frac{12}{35}$ by $\frac{8}{15}$.
8. $21\,a^{10}$ by $3\,a^7$.
9. $-63\,m^4n^6$ by $7\,m^2n^3$.
10. $5\,x^3y^2$ by $-x^3y$.
11. $12(c+d)^6$ by $3(c+d)^4$.
12. a^3b^2c by $-abc$.
13. $60(a-b)^8$ by $12(a-b)$.
14. $8(2\,m+3)$ by $4(2\,m+3)$.
15. $75\,x^5y^6$ by $-15\,x^5y^6$.
16. $81\,ab^2c$ by $27\,b^2c$.
17. $\frac{4}{5}\,x^2y$ by $\frac{1}{5}\,xy$.
18. $-\frac{5}{8}\,a^3b^2c^4$ by $-5\,ab^2$.
19. $6(a+b)^5$ by $3(a+b)^2$.

Find the numerical value when $a=2$, $b=-4$, $c=5$, and $d=-3$ of:

20. $\dfrac{7\,a+14\,b-12\,c}{13\,a-9\,b+17\,c}$.
21. $\dfrac{2\,a-b}{c-5\,d} - \dfrac{a+4\,b}{3\,c+d}$.

DIVISION OF POLYNOMIALS BY MONOMIALS

65. We have, from § 32,

$$(a+b)c = ac+bc.$$

Since the dividend is the product of the divisor and quotient (§ 60), we may regard $ac+bc$ as the dividend, c as the divisor, and $a+b$ as the quotient.

Whence, $$\frac{ac+bc}{c} = a+b.$$

Hence, to divide a polynomial by a monomial, we divide each term of the dividend by the divisor, and add the results.

Ex. Divide $9\,a^2b^2-6\,a^4c+12\,a^3bc^3$ by $-3\,a^2$.

$$\frac{9\,a^2b^2-6\,a^4c+12\,a^3bc^3}{-3\,a^2} = -3\,b^2+2\,a^2c-4\,abc^3.$$

EXERCISE 22

Divide the following:

1. $30\,a^4-25\,a^3+20\,a^2$ by $5\,a$.
2. $-33\,am^2+22\,a^4m$ by $11\,am$.
3. $18\,s^2t^3z-24\,s^2t^4z^3-12\,st^2z^2$ by $-3\,st^2z$.
4. $72\,h^3k^2x^5-60\,hk^2x^3$ by $-12\,hx^3$.
5. $\frac{5}{6}\,a^3b^2-\frac{2}{3}\,a^2b^3$ by $\frac{1}{3}\,ab$.
6. $9(a+b)^2-6(a+b)$ by $3(a+b)$.
7. $x^{m+2}+2\,x^{2m+1}-3\,x^{2m+2}$ by x^{m+1}.
8. $36\,a^{15}+28\,a^{12}-4\,a^9-20\,a^6$ by $4\,a^6$.
9. $45(a-b)^5-40(a-b)^4+25(a-b)^3$ by $5(a-b)^2$.
10. $8\,m^4n-24\,m^2n^3+12\,m^5-31\,m^2n^2$ by $6\,m^2$.
11. $(x+y)^3-9(x+y)^2+27(x+y)$ by $(x+y)$.
12. $\frac{4}{9}\,m^4-2\,m^3+\frac{9}{4}\,m^2$ by $\frac{2}{3}\,m^2$.
13. $15\,a^3-4\,a^5+8\,a^6-5\,a-2\,a^4+3\,a^2$ by $-2\,a$.
14. $a^2(2\,m+3)^3+2\,ab(2\,m+3)^2+b^2(2\,m+3)$ by $(2\,m+3)$.

DIVISION OF POLYNOMIALS BY POLYNOMIALS

66. Let it be required to divide $12+10\,x^3-11\,x-21\,x^2$ by $2\,x^2-4-3\,x$.

Arranging each expression according to the descending powers of x (§ 35), we are to find an expression which, when multiplied by the divisor, $2\,x^2-3\,x-4$, will produce the dividend, $10\,x^3-21\,x^2-11\,x+12$.

It is evident that the term containing the highest power of x in the product is the product of the terms containing the highest powers of x in the multiplicand and multiplier.

Therefore, $10\,x^3$ is the product of $2\,x^2$ and the term containing the highest power of x in the quotient.

Whence, the term containing the highest power of x in the quotient is $10\,x^3$ divided by $2\,x^2$, or $5\,x$.

Multiplying the divisor by $5\,x$, we have the product $10\,x^3-15\,x^2-20\,x$;

which, when subtracted from the dividend, leaves the remainder $-6x^2+9x+12$.

This remainder must be the product of the divisor by the rest of the quotient; therefore, to obtain the next term of the quotient, we regard $-6x^2+9x+12$ as a new dividend.

Dividing the term containing the highest power of x, $-6x^2$, by the term containing the highest power of x in the divisor, $2x^2$, we obtain -3 as the second term of the quotient.

Multiplying the divisor by -3, we have the product $-6x^2+9x+12$; which, when subtracted from the second dividend, leaves no remainder.

Hence, $5x-3$ is the required quotient.

$$\begin{array}{l|l} 10x^3-21x^2-11x+12 & 2x^2-3x-4, \text{ Divisor.} \\ \underline{10x^3-15x^2-20x} & \underline{5x-3}, \text{ Quotient.} \\ -6x^2+9x+12 & \\ \underline{-6x^2+9x+12} & \end{array}$$

The example might have been solved by arranging the dividend and divisor according to the *ascending* powers of x.

From the above example, we derive the following rule:

Arrange the dividend and divisor in the same order of powers of some common letter.

Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

Multiply the whole divisor by the first term of the quotient, and subtract the product from the dividend.

If there be a remainder, regard it as a new dividend, and proceed as before; arranging the remainder in the same order of powers as the dividend and divisor.

1. Divide $9ab^2+a^3-9b^3-5a^2b$ by $3b^2+a^2-2ab$.

Arranging according to the descending powers of a,

$$\begin{array}{l|l} a^3-5a^2b+9ab^2-9b^3 & a^2-2ab+3b^2 \\ \underline{a^3-2a^2b+3ab^2} & \underline{a-3b} \\ -3a^2b+6ab^2 & \\ \underline{-3a^2b+6ab^2-9b^3} & \end{array}$$

In the above example, the last term of the second dividend is omitted, as it is merely a repetition of the term directly above.

The work may be verified by multiplying the divisor by the quotient, which should of course give the dividend.

2. Divide $4+9x^4-28x^2$ by $-3x^2+2+4x$.

Arranging according to the ascending powers of x,

$$\begin{array}{r|l} 4-28x^2+9x^4 & 2+4x-3x^2 \\ \cline{2-2} 4+8x-6x^2 & 2-4x-3x^2 \\ \cline{1-1} -8x-22x^2+9x^4 & \\ -8x-16x^2+12x^3 & \\ \cline{1-1} -6x^3-12x^3+9x^4 & \\ -6x^3-12x^3+9x^4 & \\ \cline{1-1} \end{array}$$

EXERCISE 23

Divide the following:

1. $15x^2-11x-14$ by $3x+2$.
2. $12a^2-32a+21$ by $6a-7$.
3. $32t^2+28st-15s^2$ by $4t+5s$.
4. $c^3-8c^2-5c+84$ by $c-7$.
5. $a^2-2ab+b^2$ by $a-b$.
6. $a^2+2ab+b^2$ by $a+b$.
7. x^2+4x+4 by $x+2$.
8. x^2-6x+9 by $x-3$.

Note the form of examples 5 to 8, also the results obtained. Have you similar forms in Exercise 18?

9. $k^2-8k+15$ by $k-3$
10. h^2-h-12 by $h-4$.
11. a^3-1 by $a-1$.
12. a^3-8b^3 by $a-2b$.

Have you had multiplication problems similar to examples 9 to 12?

13. $64z^3+27d^3$ by $4z+3d$.
14. $8(b+x)^3-y^3$ by $2(b+x)-y$.
15. $x^3-5x^2y+9xy^2-9y^3$ by $x^2-2xy+3y^2$.
16. n^4-16 by $n+2$.
17. a^5+243 by $a+3$.

Do the quotients in examples 11, 12, 13, 14, 16, 17 seem to have similarity of form?

18. $16(a-b)^2-9$ by $4(a-b)+3$.
19. $\frac{8}{9}a^2-\frac{1}{3}a-\frac{3}{4}$ by $\frac{2}{3}a+\frac{1}{2}$.
20. $\frac{1}{27}g^3-\frac{1}{12}g^2k+\frac{1}{16}gk^2-\frac{1}{64}k^3$ by $\frac{1}{3}g-\frac{1}{4}k$.

21. e^4-81 by $e^3-3e^2+9e-27$.

22. a^4-256b^4 by $a-4b$. Compare example 16.

23. $13x^3+6x^4-70x^2+71x-20$ by $4+3x^2-7x$.

24. $42(c+d)^3-47(c+d)^2+17(c+d)-2$ by $7(c+d)-2$.

25. $m^5-18m^3-3m^4+24m^2+52m-21$ by $m+m^2-7$.

26. $\frac{8}{27}x^3+\frac{27}{64}$ by $\frac{2}{3}x+\frac{3}{4}$.

27. $9h^4-52h^2k^2+64k^4$ by $3h^2-2hk-8k^2$.

28. $6x^5+5x^4-57x^3-x^2+67x+28$ by $-4+3x^2-5x$.

29. $a^{2x}-b^{2n}+2b^nc^z-c^{2z}$ by $a^x+b^n-c^z$.

30. $4a^{2m+7}b^3-11a^{m+6}b^{x+1}+6a^5b^{2x-1}$ by $a^{m+2}b-2ab^{x-1}$.

67. The operation of division is often facilitated by the use of parentheses.

Ex. Divide $x^3+(a+b-c)x^2+(ab-bc-ca)x-abc$ by $x+a$.

$$\begin{array}{l|l} x^3+(a+b-c)x^2+(ab-bc-ca)x-abc & x+a \\ \underline{x^3+\quad ax^2} & \hline x^2+(b-c)x-bc \\ \quad (b-c)x^2 & \\ \quad \underline{(b-c)x^2+(ab\quad -ca)x} & \\ \qquad\qquad -\ bcx & \\ \qquad\qquad \underline{-\ bcx-abc} & \end{array}$$

EXERCISE 24

Divide the following:

1. $x^3+(a-b-c)x^2+(-ab+bc-ca)x+abc$ by $x^2+(a-b)x-ab$.

2. $x^3+(a+b-c)x^2+(ab-bc-ca)x-abc$ by $x-c$.

3. $x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$ by $x^2-(b+c)x+bc$.

4. $x^3-(a-2b-3c)x^2+(-2ab+6bc-3ca)x-6abc$ by $x^2-(a-3c)x-3ac$.

5. $x^3+(3a+b+2c)x^2+(3ab+2bc+6ca)x+6abc$ by $x+3a$.

6. $m(m+n)x^2-(m^2+n^2)x+n(m-n)$ by $mx-n$.

7. $x^2-(m-2n)x-2m^2+11mn-15n^2$ by $x+m-3n$.

8. $(2m^2+10mn)x^2+(8m^2-9mn-15n^2)x-(12mn-9n^2)$ by $2mx-3n$.

9. $x^3-(3a+2b-4c)x^2+(6ab-8bc+12ca)x-24abc$ by $x-2b$.

10. $a(a-b)x^2+(-ab+b^2+bc)x-c(b+c)$ by $(a-b)x+c$.

QUERIES

1. Is the continued product of six numbers, one half of which are positive and one half negative, a positive or a negative number? Why?

2. What three numbers are involved in these two problems:

$$(a+b)(a+b);\ (a^2+2ab+b^2)\div(a+b)?$$

3. Make a rule governing the result of $(a+b)(a+b)$.

4. Will the rule in 3 govern the result of $(x+7)(x+7)$?

5. Apply your rule to example 10, Exercise 18.

6. Find the product of $(a+b)$ and $(a-b)$ and make a general rule for such a product.

7. Can you solve example 8, Exercise 18, by this rule?

8. One of two factors of 30 is 6, what is the other? How did you find it?

9. One of two factors of x^2-x-12 is $x-4$, what is the other? Does this correspond to your definition of division?

10. Does your rule in 3 aid you in solving such problems as example 6, Exercise 23? Such forms are of frequent occurrence.

11. What is the quotient of $a^2+12a+36$ divided by $a+6$?

12. Given the multiplicand $=m$, and the product $=p$, what is the multiplier?

13. The product is x^4+x^2+1, the multiplier is x^2-x+1; find the multiplicand.

VI. INTEGRAL LINEAR EQUATIONS

68. Any term of either member of an equation (§ 3) is called a *term* of the equation.

69. A Numerical Equation is one in which all the known numbers are represented by Arabic numerals; as,

$$2x-7=x+6.$$

An Integral Equation is one each of whose members is a rational and integral expression (§ 57); as,

$$4x-5=\tfrac{2}{3}y+1.$$

70. An Identical Equation, or Identity, is an equation which is always true for specified values of the letters which enter; as, $(a+b)(a-b)\equiv a^2-b^2$.

The sign $\equiv$, read "*is identically equal to*," is frequently used in place of the sign of equality in an identity.

71. An equation is said to be *satisfied* by a set of values of certain letters involved in it when, on substituting the value of each letter in place of the letter wherever it occurs, the equation becomes identical.

Thus, the equation $x-y=5$ is satisfied by the set of values $x=8$, $y=3$; for, on substituting 8 for x, and 3 for y, the equation becomes $8-3=5$, or $5=5$; which is identical.

72. An Equation of Condition is an equation involving one or more letters, called Unknown Numbers, which is satisfied only by particular values of these letters.

Thus, the equation $x+2=5$ is not satisfied by every value of x, but only by the particular value $x=3$.

An equation of condition is usually called an *equation*.

Any letter in an equation of condition may represent an unknown number.

73. If an equation contains but one unknown number, any value of the unknown number which satisfies the equation is called a Root of the equation.

Thus, 3 is a root of the equation $x+2=5$.

To *solve* an equation is to find its roots.

74. If a rational and integral monomial (§ 57) involves a certain letter, its *degree with respect to it* is denoted by its exponent (§ 58).

If it involves two letters, its *degree with respect to them* is denoted by the sum of their exponents; etc.

Thus, $2\,ab^4x^2y^3$ is of the *second* degree with respect to x, and of the *fifth* with respect to x and y.

75. If an integral equation (§ 69) contains one or more unknown numbers, the *degree* of the equation is the degree of its term of highest degree.

Thus, if x and y represent unknown numbers,

$ax-by=c$ is an equation of the *first* degree;

$x^2+4\,x=-2$, an equation of the *second* degree;

$2\,x^2-3\,xy^2=5$, an equation of the *third* degree; etc.

A Linear, or Simple, Equation is an equation of the first degree.

PRINCIPLES USED IN SOLVING INTEGRAL EQUATIONS

76. Since the members of an equation are equal numbers, we may write the last four axioms of § 4 as follows:

1. The same number, or equal numbers, may be added to both members of an equation without destroying the equality.

2. The same number, or equal numbers, may be subtracted from both members of an equation without destroying the equality.

3. Both members of an equation may be multiplied by the same number, or equal numbers, without destroying the equality.

4. Both members of an equation may be divided by the same number, or equal numbers, without destroying the equality.

77. Transposing Terms. — Consider the equation

$$x+a-b=c.$$

Adding $-a$ and $+b$ to both members (§ 76, 1), we have

$$x=c-a+b.$$

In this case, the terms $+a$ and $-b$ are said to be *transposed* from the first member to the second.

Similarly, any term may be transposed from one member of an equation to the other by changing its sign.

78. It follows from § 77 that

If the same term occurs in both members of an equation affected with the same sign, it may be cancelled.

79. Consider the equation

$$a-x=b-c. \tag{1}$$

Multiplying each term by -1 (§ 76), we have

$$x-a=c-b;$$

which is the same as equation (1) with the sign of every term changed.

Similarly, the signs of all the terms of an equation may be changed, without destroying the equality.

80. Clearing of Fractions. — Consider the equation

$$\frac{2}{3}x-\frac{5}{4}=\frac{5}{6}x-\frac{9}{8}.$$

Multiplying each term by 24, the lowest common multiple of the denominators (Ax. 7, § 4), we have

$$16\,x-30=20\,x-27,$$

where the denominators have been removed.

Removing the fractions from an equation by multiplication is called *clearing the equation of fractions.*

SOLUTION OF INTEGRAL LINEAR EQUATIONS

81. To solve an equation involving one unknown number, we put it into a succession of forms, which finally leads to the value of the root.

This process is called *transforming* the equation.

Every transformation is effected by means of the principles of §§ 76 to 80, inclusive.

82. Examples.

1. Solve the equation $5x-7=3x+1$.

Transposing $3x$ to the first member, and -7 to the second (§ 77), we have

$$5x-3x=7+1.$$

Uniting similar terms, $2x=8.$

Dividing both members by 2 (§ 76, 4),

$$x=4.$$

To *verify* the result, put $x=4$ in the given equation.

Thus, $20-7=12+1$; which is identical.

2. Solve the equation

$$\frac{7}{6}t-\frac{5}{3}=\frac{3}{5}t-\frac{1}{4}.$$

Clearing of fractions by multiplying each term by 60, the L. C. M. of 6, 3, 5, and 4, we have

$$70t-100=36t-15.$$

Transposing $36t$ to the first member, and -100 to the second,

$$70t-36t=100-15.$$

Uniting terms, $34t=85.$

Dividing by 34, $$t=\frac{85}{34}=\frac{5}{2}.$$

Verify this result by substituting $t=\frac{5}{2}$ in the given equation.

3. Solve the equation

$$(5-3x)(3+4x)=62-(7-3x)(1-4x).$$

Expanding, $15+11x-12x^2=62-(7-31x+12x^2).$

Or, $15+11x-12x^2=62-7+31x-12x^2.$

Cancelling the $-12x^2$ terms (§ 78), and transposing,

$$11x-31x=62-7-15.$$

Uniting terms, $-20x=40.$

Dividing by -20, $x=-2.$

Verify the result by substituting $x=-2$ in the given equation.

To *expand* an algebraic expression is to perform the operations indicated.

From these examples, we have the following rule for solving an integral linear equation with one unknown number:

Clear the equation of fractions, if any, by multiplying each term by the L. C. M. of the denominators of the fractional terms.

Remove the parentheses, if any, by performing all the operations indicated.

Transpose the unknown terms to the first member, and the known to the second; cancelling any term which has the same coefficient in both members.

Unite similar terms, and divide both members by the coefficient of the unknown number.

The pupil should verify every result.

EXERCISE 25

Solve the following equations, in each case verifying the result:

1. $5x+13=28$.
2. $7z=4z-33$.
3. $11n+71=6n+76$.
4. $8d-2=5d-26$.
5. $15x+19=11x-5$.
6. $13-21k=34-14k$.
7. $25x-3=4+18x$.
8. $24t-28=14t-48$.
9. $26-4x=31-2x$.
10. $16R-47=8R-43$.
11. $17+14x=11x+16$.
12. $43k-27=37-149k$.
13. $12y+15=15y+17$.
14. $98-16x=23-41x$.
15. $29x-8+17=32x-14x-24$.
16. $35z-41=-81+63z-58z$.
17. $0=31t-14t+3t+30$.
18. $\frac{1}{3}m+\frac{1}{2}=2-\frac{5}{6}-\frac{1}{2}m$.
19. $\frac{2}{3}v-\frac{1}{6}=\frac{5}{6}v+\frac{1}{3}$.

20. $\frac{5}{6}k+\frac{1}{6}k+\frac{8}{3}k=\frac{11}{3}$.

21. $\frac{5}{6}s=\frac{7}{4}s-\frac{3}{8}s-\frac{13}{48}$.

22. $\frac{4}{7}q+\frac{5}{4}=\frac{9}{14}q+\frac{3}{28}q$.

23. $\frac{2}{5}x-\frac{38}{3}=\frac{8}{9}x-\frac{4}{3}x$.

24. $\frac{2}{3}v-\frac{4}{5}v=\frac{7}{8}v-\frac{1}{20}v-\frac{23}{24}$.

25. $5(x+3)-7=6(2x-3)+40$.

26. $12k-(4k-7)=3k-(9k-28)$.

27. $75-8(7u+5)=6u-(4u+52)$.

28. $5R-3(2-8R)=9R-4(1-4R)$.

29. $(4-3z)(5+4z)=(8+2z)(1-6z)-82$.

30. $\frac{1}{3}(4x+1)+\frac{1}{5}(6x-2)-\frac{1}{6}(5x+8)=2$.

PROBLEMS LEADING TO INTEGRAL LINEAR EQUATIONS WITH ONE UNKNOWN NUMBER

83. For the solution of a problem by algebraic methods, the following suggestions will be found of service:

1. Represent the unknown number, or one of the unknown numbers if there are several, by *some* letter, as x.

2. Every problem contains, explicitly or implicitly, *at least as many distinct statements as there are unknown numbers involved.* Use all but one of these to express the other unknown numbers in terms of x.

3. Use the remaining statement to form an equation.

84. Problems.

1. Divide 45 into two parts such that the less part shall be one-fourth the greater.

Here there are *two* unknown numbers; the greater part and the less.

In accordance with the first suggestion of § 83, we represent the greater part by x.

The first statement of the problem is, implicitly:

The sum of the greater part and the less is 45.

The second statement is:

The less part is one-fourth the greater.

In accordance with the second suggestion of § 83, we use the *first statement* to express the less part in terms of x.

Thus, the less part is represented by $45-x$.

We now, in accordance with the third suggestion, use the *second statement* to form an equation.

Thus, $45-x=\frac{1}{4}x$.

Clearing of fractions, $180-4x=x$.

Transposing, $-4x-x=-180$, or $-5x=-180$.

Dividing by -5, $x=36$, the greater part.

Then, $45-x=9$, the less part.

Verify by substituting $x=36$ in the given equation.

2. A had twice as much money as B; but after giving B \$28, he has $\frac{2}{3}$ as much as B. How much had each at first?

Let x represent the number of dollars B had at first.

Then, $2x$ will represent the number A had at first.

Now after giving B \$28, A has $2x-28$ dollars, and B, $x+28$ dollars; we then have the equation

$$2x-28=\frac{2}{3}(x+28).$$

Clearing of fractions, $6x-84=2(x+28)$.

Expanding, $6x-84=2x+56$.

Transposing, $4x=140$.

Dividing by 4, $x=35$, the number of dollars B had at first;

and $2x=70$, the number of dollars A had at first.

Verify the result.

3. A is 3 times as old as B, and 8 years ago he was 7 times as old as B. Required their ages at present.

Let $n=$the number of years in B's age.

Then, $3n=$the number of years in A's age.

Also, $n-8=$the number of years in B's age 8 years ago,

and $3n-8=$the number of years in A's age 8 years ago.

But A's age 8 years ago was 7 times B's age 8 years ago.

Whence, $3n-8=7(n-8)$.

Expanding, $3n-8=7n-56$.

Transposing, $-4n=-48$.

Dividing by -4, $n=12$, the number of years in B's age.

Whence, $3n=36$, the number of years in A's age.

Verify the result.

4. A sum of money amounting to $4.32 consists of 108 coins, all dimes and cents; how many are there of each kind?

Let x = the number of dimes.

Then, $108 - x$ = the number of cents.

Also, the x dimes are worth $10\,x$ cents.

But the entire sum amounts to 432 *cents*.

Whence, $10\,x + 108 - x = 432$.

Transposing, $9\,x = 324$.

Whence, $x = 36$, the number of dimes;

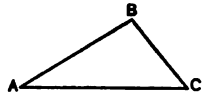
and $108 - x = 72$, the number of cents.

Verify the result.

EXERCISE 26

1. The difference of two numbers is 12, and 7 times the smaller exceeds the greater by 30. Find the numbers.

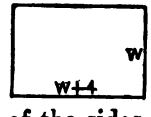
2. The sum of two sides, AB and BC, of the triangle ABC is 23, and the lesser side exceeds their difference by 7. Find the sides AB and BC. Can more than one such triangle be drawn?



3. Find two numbers whose sum is $\frac{7}{8}$, and difference $\frac{1}{8}$.

4. The sum of two numbers is 35, and their difference is three-fifths the larger number. Find the numbers.

5. A is 5 years older than B, and the sum of their ages is 39 years. How old is each?



6. A rectangle is 4 feet longer than it is wide. If 4 feet were added to the length the area would be increased 40 square feet. Find the length of the sides.

7. A rectangle is 6 feet longer than it is wide. If 3 feet be added to its width and 4 feet be subtracted from its length its area will not be changed. Find the length and breadth.

8. A man counting the coins he has in his hand finds that he has three times as many quarters as half dollars, five

more dimes than quarters, and twice as many five-cent pieces as dimes. The entire sum of money is $2.85. How many coins of each kind?

9. A is 25 years of age, B is 9 years of age. In how many years will A be twice as old as B?

10. The length of a rectangle is 5 feet more than the width. If 4 feet be taken from the length and 4 feet from the width the area of the rectangle will be diminished 124 square feet. Find the length and breadth of the rectangle.

11. A certain number of two digits is equal to 9 times the sum of the digits and the digit in ten's place is 7 greater than the digit in unit's place. Find the number.

12. Divide $300 among A, B, and C so that $\frac{1}{3}$ of B's share plus $20 may equal A's share, and C and B may have equal amounts.

13. A man has $4.10, all five-cent and fifty-cent pieces; and he has 5 more five-cent pieces than fifty-cent pieces. How many has he of each?

14. The difference between $\frac{3}{4}$ and $\frac{1}{3}$ of a certain number exceeds $\frac{1}{5}$ of it by 44. What is the number?

15. A has $5.50 and B $3.50; how much money must A give B in order that B may have $\frac{4}{5}$ as much as A?

16. A room is $\frac{3}{5}$ as long as it is wide; if the length were diminished 3 feet and the width increased by the same amount, the room would be square. Find its dimensions.

Note: Oranges come packed in boxes, a box containing 86, 90, 110, 126, 150, 175 oranges. A box marked 90's indicates that there are 90 oranges in that box.

17. A merchant buys oranges, 150's, a certain number of boxes at $3.25, twice as many at $3.00 and six boxes at $3.50, paying $39.50 for the entire lot. Find the average cost per dozen oranges.

18. A merchant buys oranges, 90's, some at \$2.00 per box, $\frac{2}{5}$ as many at \$2.20 per box, paying \$28.80 for the entire lot. Can he make a profit retailing them at 29¢ per dozen, no allowance being made for expense of handling?

Note: Banana dealers estimate the value of a *bunch* of bananas by the number of *hands* on a bunch. A *hand* is a cluster of bananas grouped together and contains 12 to 16 bananas.

19. A merchant bought three lots of bananas; some, 8 hands, at 85¢; three times as many 12 hands at \$1.15, and 5 bunches, 10 hands, at \$1.05, paying \$18.15 for all. Find the approximate average cost per dozen bananas, averaging 15 bananas to the hand.

20. A given square has 39 square feet more area than a given rectangle. The length of the rectangle is 3 feet more than a side of the square, and the breadth of the rectangle is 5 feet less than a side of the square. Find the dimensions of each figure.

21. Divide \$480 among A, B, C, and D so that B shall have twice as much as A, B shall have \$6 more than C, and C and D together as much as A and B together.

22. Find two numbers whose difference is 17, such that the square of the greater exceeds the square of the less by 1037.

23. A room is $\frac{3}{2}$ as long as it is wide, and 60 feet of picture molding are required to go around it. Find the number of square feet in the floor.

24. A starts to walk from Boston to Rockland, 19 miles, at the same time, B starts to walk from Rockland to Boston. A walks $\frac{1}{2}$ mile an hour faster than B. They meet in $3\frac{1}{2}$ hours. Find the rate of each.

(Let R = number of miles per hour A walks.)

25. The sum of \$900 is invested, part at 4%, and the rest at 5%, per annum, and the total annual income is \$42. How much is invested in each way?

26. In 9 years B will be $\frac{5}{6}$ as old as A; and 12 years ago he was $\frac{3}{5}$ as old. What are their ages?

(Let n represent the number of years in A's age 12 years ago.)

27. A man buys irrigated farm land, some at \$17 per acre, and three times as much less 160 acres at \$15 per acre, paying \$17,440 for the entire farm. He also pays \$2.50 an acre for a water right. He sells the land for \$21 per acre. What is his profit?

28. A has $\frac{1}{5}$ of a certain sum of money, B has $\frac{2}{7}$, C \$5 less than $\frac{3}{8}$, D the balance which is \$44. Find C's share.

29. Find three consecutive numbers such that the square of the greatest exceeds the product of the other two by 70.

30. Find three consecutive numbers such that if the square of the least number be subtracted from the product of the other two the remainder will be 47.

31. A number consists of two digits, and the ten's digit is 5 greater than the unit's digit. The difference between the squares of the digits is 65. What is the number?

32. A is 10 years older than B; 4 years ago B was $\frac{2}{3}$ as old as A will be in 5 years. Find the age of each.

33. There are two heaps of coins, the first containing 5-cent pieces, and the second 10-cent pieces. The second heap is worth 20 cents more than the first, and has 8 fewer coins. Find the number in each heap.

34. A certain number is composed of two digits; the number is six more than six times the sum of the digits, and the digit in unit's place is $\frac{2}{3}$ the digit in ten's place. Find the number.

35. Find four consecutive odd numbers such that the product of the first and third shall be less than the product of the second and fourth by 64.

VII. PRODUCTS AND FACTORS

85. A Power of a Power. — Required the value of $(a^2)^3$.

By § 6, $(a^2)^3 = a^2 \times a^2 \times a^2 = a^6$.

The general case: — Required the value of $(a^m)^n$, where m and n are any positive integers.

We have, $(a^m)^n = a^m \times a^m \times \cdots$ to n factors

$= a^{m+m+\cdots \text{ to } n \text{ terms}} = a^{mn}$.

86. A Power of a Product. — Required the value of $(ab)^3$.

By § 6, $(ab)^3 = ab \times ab \times ab = a^3b^3$.

The general case: — Required the value of $(ab)^n$, where n is any positive integer.

We have, $(ab)^n = ab \times ab \times \cdots$ to n factors $= a^n b^n$.

In like manner, $(abc\cdots)^n = a^n b^n c^n \cdots$,

whatever the number of factors in $abc\cdots$.

87. A Power of a Monomial.

1. Find the value of $(-5a^4)^3$.

By § 26, $(-5a^4)^3 = [(-5) \times a^4]^3$

$= (-5)^3 \times (a^4)^3$ (§ 86) $= -125\, a^{12}$ (§ 85).

2. Find the value of $(-2m^3n)^4$.

We have, $(-2m^3n)^4 = (-2)^4 \times (m^3)^4 \times n^4 = 16\, m^{12} n^4$.

88. From §§ 85 and 86 and the examples of § 87, we have the following rule for raising a rational and integral monomial (§ 57) to any power whose exponent is a positive integer.

Raise the absolute value of the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

Give to every power of a positive term, and to every *even* power of a negative term, the positive sign; and to every *odd* power of a negative term the negative sign.

EXERCISE 27

Expand the following:

1. $(x^3y^4z^5)^8$.
2. $(m^6n^2p)^{11}$.
3. $(-ab^7c^{10})^7$.
4. $(-11\,x^9y^8)^2$.
5. $(7\,a^mb^{2n})^3$.
6. $(-n^3x^5y^4)^{10}$.
7. $(2\,m^6x^7)^6$.
8. $(-4\,x^2yz^{11})^4$.
9. $(a^nb^3c)^p$.
10. $(x^{5m}y^{4n}z^{2p})^{12}$.
11. $(-3\,m^2n^9x^6)^5$.
12. $(-2\,am^5n^7)^9$.

Find the factors of the following:

13. $25\,a^2b^4$.
14. $32\,m^5$.
15. $48\,a^4b^2c$.
16. $21\,a^7$.
17. $3\,a^3$.
18. a^{2n}.
19. a^{n+1}.
20. a^{2n+1}.
21. $343\,a^3c^4$.
22. $243\,m^3z^6$.
23. $165\,a^4z^2$.
24. $-282\,a^{2m}c^{2x}$.

89. **Type I. Product of the Sum and Difference of Two Numbers.** — Let it be required to multiply $a+b$ by $a-b$.

$$\begin{array}{r} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \qquad -b^2 \end{array}$$

Whence, $(a+b)(a-b)=a^2-b^2$.

That is, the product of the sum and difference of two numbers equals the difference of their squares.

1. Multiply $6\,a+5\,b^3$ by $6\,a-5b^3$.

By the rule,

$$(6\,a+5\,b^3)(6\,a-5\,b^3)=(6\,a)^2-(5\,b^3)^2=36\,a^2-25\,b^6.$$

2. Multiply $-x^2+4$ by $-x^2-4$.

$$\begin{aligned}(-x^2+4)(-x^2-4)&=[(-x^2)+4][(-x^2)-4]\\&=(-x^2)^2-4^2=x^4-16.\end{aligned}$$

EXERCISE 28

Expand the following:

1. $(4\,a+3\,b)(4\,a-3\,b)$.
2. $(2\,x+4\,y)(2\,x-4\,y)$.
3. $(3\,c+8)(3\,c-8)$.
4. $(8\,R+1)(8\,R-1)$.

5. $(6d+5t)(6d-5t)$.
6. $(11k^4+3l^3)(11k^4-3l^3)$.
7. $(9g+2)(9g-2)$.
8. $(15a+13b)(15a-13b)$.
9. $(-x+7)(x+7)$.

(Prove this last result by actual multiplication.)

10. $(12x+y)(12x-y)$.
11. $(-19c^2+4d^4)(-19c^2-4d^4)$.
12. From what factors do you obtain x^2-9?
13. From what do you obtain $4a^2-25$?
14. Find the factors of $9c^2-49d^2$.

By reversing the product rule in § 89, this rule follows:

To factor the difference of two squares, extract the square root of the first square, and of the second square; add the results for one factor, and subtract the second result from the first for the other factor.

Note: It is not always possible to factor an expression; there are, however, certain forms which can always be factored; these will be considered in the present work.

Factor the following:

(*Check:* If results are correct, the product of the factors will equal the given expression.)

15. $9a^2-4b^2$.
16. $36m^2-25k^2$.
17. a^2-9c^2.
18. $25e^2-81h^2$.
19. $49-4d^2$.
20. $36x^4-121y^6$.
21. $16-25a^8$.
22. $100m^4x^{12}-169y^{10}$.
23. $(m+n)^2-z^2$.
24. $49a^8-144b^6$.
25. $(x+y)^2-(a+b)^2$.
26. $(x+y)^2-(a-b)^2$.
27. $(2a+x)^2-(a-2x)^2$.

Expand the following:

28. $(5a^2+12b^3c)(5a^2-12b^3c)$.
29. $(a^x+e^y)(a^x-e^y)$.
30. $(6m+4b^2)(6m-4b^2)$.
31. $(c^{3x}+d^{3y})(c^{3x}-d^{3y})$.

Sometimes the factors of an expression admit of further factoring:

32. $x^4-81=(x^2+9)(x^2-9)$ (The second factor can be factored)
$=(x^2+9)(x+3)(x-3)$.

33. Factor $16\,m^4-625\,y^4$. 34. Factor a^8-b^8.

35. Factor $81\,c^8-16\,d^4$.

36. Factor $(R+S+3)^2-(R-S-4)^2$.

37. Does $36\,c^2-2$ belong to this type?

38. Can you factor x^4+9 by this type?

90. By division:

$$\frac{a^2-b^2}{a+b}=a-b, \quad (1) \qquad \frac{a^2-b^2}{a-b}=a+b. \quad (2)$$

(Compare Exercise 23, Ex. 18.)

1. Divide $25\,y^2z^4-9$ by $5\,yz^2-3$.

By § 88, $25\,y^2z^4$ is the square of $5\,yz^2$; then by (2),

$$\frac{25\,y^2z^4-9}{5\,yz^2-3}=5\,yz^2+3.$$

2. Divide $x^2-(y-z)^2$ by $x+(y-z)$.

By (1), $$\frac{x^2-(y-z)^2}{x+(y-z)}=x-(y-z)=x-y+z.$$

EXERCISE 29

Find, without actual division, the values of the following:

1. $\dfrac{a^2-4}{a+2}$. 3. $\dfrac{25\,n^4-1}{5\,n^2-1}$. 5. $\dfrac{1-144\,a^{2x}b^{2c}}{1-12\,a^xb^c}$.

2. $\dfrac{x^2-9}{x-3}$. 4. $\dfrac{64\,n^8-x^{10}}{8\,n^4+x^5}$. 6. $\dfrac{49\,x^4z^2-64}{7\,x^2z+8}$.

91. **Type II. Square of a Binomial.** — Let it be required to square $a+b$.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+\ ab \\ \quad\ ab+b^2 \\ \hline \end{array}$$

Whence, $$(a+b)^2=a^2+2ab+b^2. \quad (1)$$

That is, the square of the sum of two numbers equals the square of the first, plus twice the product of the first by the second, plus the square of the second.

1. Square $3a+2b$.

We have, $(3a+2b)^2=(3a)^2+2(3a)(2b)+(2b)^2$
$=9a^2+12ab+4b^2$.

Let it be required to square $a-b$.

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2- \quad ab \\ \quad -ab+b^2 \\ \hline \end{array}$$

Whence, $(a-b)^2=a^2-2ab+b^2$. (2)

That is, the square of the difference of two numbers equals the square of the first, minus twice the product of the first by the second, plus the square of the second.

In the remainder of the work we shall use the expression "the *difference* between a and b" to denote the *remainder obtained by subtracting b from a.*

The result (2) may also be derived by substituting $-b$ for b, in equation (1).

2. Square $4x^2-5$.

We have, $(4x^2-5)^2=(4x^2)^2-2(4x^2)(5)+5^2$
$=16x^4-40x^2+25$.

If the first term of the binomial is *negative*, it should be written, negative sign and all, in parenthesis, before applying the rules.

3. Square $-2a^3+9$.

We have, $(-2a^3+9)^2=[(-2a^3)+9]^2$
$=(-2a^3)^2+2(-2a^3)(9)+9^2$
$=4a^6-36a^3+81$.

EXERCISE 30

The following 18 examples are for mental drill:

1. $(x+3)^2$.
2. $(a-4)^2$.
3. $(c+9)^2$.
4. $(2x+7)^2$.
5. $(4y-6z)^2$.
6. $(3ac-4b)^2$.
7. $(x+4)^2$.
8. $(-4k+3d)^2$.
9. $(h-11)^2$.
10. $(v-12w)^2$.
11. $(4a+13b)^2$.
12. $(15x-1)^2$.

Note that in each of these trinomial squares, the first and third terms are perfect squares and positive, and the middle term is twice the product of the square roots of the first and third terms.

What sign does the middle term have?

In each of the following expressions supply the missing term which will form a perfect trinomial square:

13. x^2+4x. 15. c^2+16. 17. b^2-4b.

14. a^2+9. 16. x^2+12x. 18. s^2+4.

Can you substitute other numbers than those you used and still form a perfect square?

19. $x^2+10x+25$ is the square of what?

20. x^2-6x+9 is composed of what factors?

(Compare example 1.)

21. Factor $x^2+2xy+y^2$.

22. From what two factors do you obtain $16a^2+8a+1$?

By reversing the product rule in § 91, this rule follows:

To factor a trinomial square, extract the square roots of the first and third terms, and connect the results by the sign of the second term. This gives one of the equal factors.

23. Factor $4x^2+12xy+9y^2$. 26. Factor $25k^2+60hk+36h^2$.

24. Factor $9y^2+6y+1$. 27. Expand $(3x+2y)^2$.

25. Factor $c^2+8c+16$. 28. Expand $(8x^5+9x^6)^2$.

29. From what do you obtain $x^2y^2+14xy+49$?

Sometimes the factors of an expression admit of further factoring:

30. $x^4-8x^2+16=(x^2-4)(x^2-4)$

$=(x+2)(x-2)(x+2)(x-2)$ [by § 89].

This result may be written $(x+2)^2(x-2)^2$.

31. Factor a^4-18a^2+81.

32. Factor $49t^2+168tu+144u^2$.

33. Factor $25(a+b)^2+40(a+b)c+16c^2$.

34. Factor $16\,m^4-72\,m^2v^2+81\,v^4$.

35. Expand $(x+y+z)(x-y+z)$.

$$\begin{aligned}(x+y+z)(x-y+z) &= [(x+z)+y]\,[(x+z)-y] \\ &= (x+z)^2-y^2 \\ &= x^2+2\,xz+z^2-y^2.\end{aligned}$$

36. Expand $(a+b-c)(a-b+c)$.

By § 46, $$\begin{aligned}(a+b-c)(a-b+c) &= [a+(b-c)]\,[a-(b-c)] \\ &= a^2-(b-c)^2, \text{ by the rule,} \\ &= a^2-(b^2-2\,bc+c^2) \\ &= a^2-b^2+2\,bc-c^2.\end{aligned}$$

37. $(x^2+x+1)(x^2+x-1)$.

38. $(a^2+1+3\,a)(a^2+1-3\,a)$.

39. $(x+y+3)(x-y-3)$.

40. $(a^2+5\,a-4)(a^2-5\,a+4)$.

41. Factor $a^2+2\,ab+b^2-c^2$.

$$\begin{aligned}&= (a+b)^2-c^2 \\ &= (a+b+c)(a+b-c).\end{aligned}$$

42. Factor $a^2+6\,a+9-4\,c^2$.

43. Factor $9-a^2+2\,ab-b^2$ (§ 46).

44. Factor $a^2+2\,ab+b^2-c^2-2\,cd-d^2$.

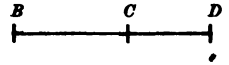
45. Factor $a^2-4\,ax+4\,x^2-b^2+6\,by-9\,y^2$.

46. Factor $x^2-y^2-2\,yz-z^2$.

47. Is $x^2-8\,x+25$ a perfect square? Why?

48. Square both members of the equation

$(BC)=(BD)-(CD)$. (See figure.)



92. Type III. Product of Two Binomials having the Same First Term. — Let it be required to multiply $x+a$ by $x+b$.

$$\begin{array}{rr} x+a & \\ x+b & \\ \hline x^2+ & ax \\ + & bx+ab \\ \hline \end{array}$$

Whence, $(x+a)(x+b)=x^2+(a+b)x+ab$.

That is, the product of two binomials having the same first term equals the square of the first term, plus the algebraic sum of the second terms multiplied by the first term, plus the product of the second terms.

1. Multiply $x-5$ by $x+3$.

The coefficient of x is the sum of -5 and $+3$, or -2.
The last term is the product of -5 and $+3$, or -15.
Whence, $(x-5)(x+3)=x^2-2x-15$.

2. Multiply $x-5$ by $x-3$.

The coefficient of x is the sum of -5 and -3, or -8.
The last term is the product of -5 and -3, or 15.
Whence, $(x-5)(x-3)=x^2-8x+15$.

3. Multiply $ab-4$ by $ab+7$.

The coefficient of ab is the sum of -4 and 7, or 3.
The last term is the product of -4 and 7, or -28.
Whence, $(ab-4)(ab+7)=a^2b^2+3ab-28$.

4. Multiply x^2+6y^2 by x^2+8y^2.

The coefficient of x^2 is the sum of $6y^2$ and $8y^2$, or $14y^2$.
The last term is the product of $6y^2$ and $8y^2$, or $48y^4$.
Whence, $(x^2+6y^2)(x^2+8y^2)=x^4+14x^2y^2+48y^4$.

EXERCISE 31

Expand the following by inspection:

1. $(x+2)(x+3)$.
2. $(x-3)(x+7)$.
3. $(x-12)(x-1)$.
4. $(x-9)(x+2)$.
5. $(z^2+13)(z^2+2)$.
6. $(a^3-1)(a^3+27)$.
7. $(c^5-4)(c^5+6)$
8. $(a^x+3)(a^x+9)$.
9. $(R+2C)(R+9C)$.
10. $(e-8y)(e-9y)$.
11. $(a^3+2)(a^3-5)$.
12. $(x^n-1)(x^n+7)$.
13. $(x^2-20)(x^2+4)$.
14. $(e^z+3)(e^z-11)$.
15. From what factors do you obtain $x^2+8x+15$?

(Compare example 9, Exercise 23.)

16. What are the factors of $x^2+7x+12$?

17. **Factor** $x^2+4x-12$.

By the rule in § 92, the product takes the form

$$x^2+ax+b.$$

To factor a trinomial of the form

$$x^2+ax+b,$$

reverse this process.

Hence, to obtain the second terms of the binomials reverse the rule for products and find two numbers whose algebraic sum is the coefficient of x, and whose product is the last term of the trinomial. The numbers may be found by inspection.

18. **Factor** $x^2+14x+45$.

We find two numbers whose sum is 14 and product 45.

By inspection, we determine that these numbers are 9 and 5.

Whence, $x^2+14x+45=(x+9)(x+5)$.

Factor the following:

19. $x^2+3x-10$.
20. $x^2-12x+11$.
21. $x^2-5x-14$.
22. a^4+16a^2+15.
23. $m^2+5m-24$.
24. c^2-c-72.
25. $d^2+37d+36$.
26. k^4+5k^2-14.
27. R^6-13R^3+22.
28. $m^2+6m-16$.
29. $1+2a-99a^2$.
30. $a^2+18a+56$.
31. $c^2-10c-75$.
32. $k^2-6k-72$.
33. $m^2+27m+72$.
34. $a^2+17a+72$.
35. $(x-y)^2-9(x-y)+20$.
36. $(a+b)^2+(a+b)-56$.
37. $(c+d)^2-4(c+d)-60$.

Expand by inspection:

38. $(a^2-8)(a^2+12)$.
39. $(c+7)(c+7)$.
40. $(h+3)(h+3)$.
41. $[(x+y)+2][(x+y)-14]$.
42. $[(m+R)-8][(m+R)+6]$.

Find numbers which will make the following factorable:

43. $x^2+(?)x+36$.
44. $a^2\ (\)a-72$.
45. $c^2\ (\)c-48$.

EXERCISE 32

Select the type to which each of the following belongs and then factor:

1. x^4-4x^2-32.
2. $a^2+8a+16$.
3. $a^2+17a+16$.
4. $a^2+10a+16$.
5. $k^2-12k+36$.
6. x^2+2x+1.
7. $36x^2-9y^2$.
8. $a^2-16+2ab+b^2$.
9. a^4-625.
10. $z^2-z-132$.
11. $m^2-50m+49$.
12. $m^2-14m+49$.
13. Can you factor x^2+x+1 by any type you have had?

The accuracy of your factors can always be proved by finding the product of your factors.

14. Factor $(x+y)^2-11(x+y)+30$.
15. Factor $x^2+(2m+3k)x+6mk$.

93. Type IV. Product of Two Binomials of the Form $mx+n$ and $px+q$. — We find by multiplication:

$$\begin{array}{l} mx+n \\ \times \\ px+q \\ \hline mpx^2+\quad npx \\ \quad +\quad mqx+nq \\ \hline mpx^2+(np+mq)x+nq \end{array}$$

The first term of this result, mpx^2, is the product of the first terms of the binomial factors, and the last term, nq, the product of the second terms.

The middle term, $(np+mq)x$, is the sum of the products of the terms, in the binomial factors, connected by cross lines.

Ex. Multiply $3x+4$ by $2x-5$.

The first term is the product of $3x$ and $2x$, or $6x^2$.

The coefficient of x is the sum of 4×2 and $3\times(-5)$; that is, $8-15$, or -7.

The last term is the product of 4 and -5, or -20.

Whence, $(3x+4)(2x-5)=6x^2-7x-20$.

EXERCISE 33

Expand the following by inspection:

1. $(x+2)(4x+3)$.
2. $(3x-2)(2x+1)$.
3. $(2x-7)(5x+3)$.
4. $(8x-1)(7x+2)$.
5. $(a-6)(3a-4)$.
6. $(2k+15)(4k-11)$.
7. $(6e-5)(4e-3)$.
8. $(2d-1)(5d+2)$.
9. $(3m+4x)(2m-3x)$.
10. $(2a^2+3y)(3a^2+5y)$.
11. $(6a^2+x^2)(8a^2-5x^2)$.
12. $(5R-4H)(3R+11H)$
13. $(m+11b)(11m+b)$.
14. $(6k-5l)(5k+6l)$.

94. Note that the product of two factors of the above form is a trinomial of the form

(Type IV.) $$ax^2+bx+c.$$

To factor a trinomial of the form

$$ax^2+bx+c,$$

reverse the above process. Hence,

To resolve a trinomial of the form ax^2+bx+c into two binomial factors, the first terms of the binomials must be such that their product is ax^2; the second terms must be such that their product is c; the sum of the cross products must be bx.

1. Factor $3x^2+8x+4$.

The first terms of the binomial factors must be such that their product is $3x^2$; they can be only $3x$ and x.

The second terms must be such that their product is 4.

The numbers whose product is 4 are 4 and 1, -4 and -1, 2 and 2, and -2 and -2; the possible cases are represented below:

$$\begin{array}{ccc} x+4 & x+1 & x-4 \\ \times & \times & \times \\ \underline{3x+1} & \underline{3x+4} & \underline{3x-1} \\ 13x & 7x & -13x \end{array}$$

$$\begin{array}{ccc} x-1 & x+2 & x-2 \\ \times & \times & \times \\ \underline{3x-4} & \underline{3x+2} & \underline{3x-2} \\ -7x & 8x & -8x \end{array}$$

The corresponding middle term of the trinomial, obtained by cross-multiplication, as in § 93, is given in each case; and only the factors $x+2$, $3x+2$ satisfy the condition that the middle term shall be $8x$.

Then, $3x^2+8x+4=(x+2)(3x+2)$.

2. Factor $6x^2-x-2$.

The first terms of the factors must be $6x$ and x, or $3x$ and $2x$.

The second terms must be 2 and -1, or -2 and 1.

The possible cases are given below:

$6x+2$	$6x-1$	$6x-2$	$6x+1$
$x-1$	$x+2$	$x+1$	$x-2$
$-4x$	$11x$	$4x$	$-11x$
$3x+2$	$3x-1$	$3x-2$	$3x+1$
$2x-1$	$2x+2$	$2x+1$	$2x-2$
x	$4x$	$-x$	$-4x$

Only the factors $3x-2$ and $2x+1$ satisfy the condition that the middle term shall be $-x$.

Then, $6x^2-x-2=(3x-2)(2x+1)$.

The following suggestions will be found of service:

(*a*) If the last term of the trinomial is positive, the last terms of the factors will be both +, or both −, according as the middle term of the trinomial is + or −.

Thus, in Ex. 1, we need not have tried the numbers -1 and -4, nor -2 and -2; this would have left only three cases to consider.

(*b*) If the last term of the trinomial is negative, the last terms of the factors will be one +, the other −.

If the x^2 term is negative, the entire expression should be enclosed in parentheses preceded by a − sign.

If the coefficient of x^2 is a perfect square, and the coefficient of x divisible by the square root of the coefficient of x^2, the expression may be readily factored by the method of § 91.

3. Factor $9x^2-18x+5$.

In this case, 18 is divisible by the square root of 9.

We have $9x^2-18x+5=(3x)^2-6(3x)+5$.

We find two numbers whose sum is -6, and product 5.

The numbers are -5 and -1.

Then, $9x^2-18x+5=(3x-5)(3x-1)$.

EXERCISE 34

Factor the following by inspection:

1. $3x^2+20x+12$.
2. $14x^2+5x-1$.
3. $8x^2-14x-15$.
4. $20a^2-27a+9$.
5. $16m^2+16m+3$.
6. $15R^2+4R-4$.
7. $22a^4-19a^2+4$.
8. $30c^8+41c^4+6$.
9. $10a^2x^2-3ax-18$.
10. $30x^2+17dx-2d^2$.
11. $36x^2-19xy-6y^2$.
12. $49a^2-42ab+8b^2$.
13. $54a^{2x}+15a^xy+y^2$.
14. $48a^4-22a^2x^2-5x^4$.
15. $50t^2-55st+14s^2$.
16. $72c^2d^2-13abcd-15a^2b^2$.

EXERCISE 35

Select the type to which each of the following belongs and then factor:

1. $9b^2-20bc+4c^2$.
2. $9b^2-12bc+4c^2$.
3. $9b^2-4c^2$.
4. $(235)^2-(234)^2$.
5. $9b^2-16bc-4c^2$.
6. $k^2-13ky-48y^2$.
7. $k^2+14ky+49y^2$.
8. $15c^2-19cd-56d^2$.
9. $6x^2-7x-20$.
10. $a^2-16ab+64b^2$.
11. $36d^2x^2-36dmx+9m^2$.
12. $256a^4-800a^2b^2+625b^4$.
13. x^8-y^8.
14. Can you factor $3x^2-2x+12$?

Solve the following equations and verify each result:

15. $(x+3)^2+(x+5)(3x-4)=(2x+5)^2$.
16. $(3t+5)(3t-5)-(t+7)(t-1)=(8t+3)(t-1)$.
17. $(2m-3)^2+(m+8)(m-8)=(5m-1)(m+3)$.

95. It is not possible to factor every expression of the form x^2+ax+b by the method of § 92.

Thus, let it be required to factor $x^2+18x+35$.

We must find two numbers whose sum is 18, and product 35.

The only pairs of positive integral factors of 35 are 7 and 5, and 35 and 1; and in neither case is the sum 18.

It is also impossible to factor every expression of the form ax^2+bx+c by the method of § 94.

Thus, it is impossible to find two binomial factors of the expression $4x^2+4x-1$ by the method of § 94.

In § 236 will be given a *general* method for the factoring of any expression of the form x^2+ax+b, or ax^2+bx+c.

96. Type V. When the expression is in the form

$$x^4+ax^2y^2+y^4.$$

Certain trinomials of the above form may be factored by expressing them as the difference of two perfect squares, and then employing § 89.

1. Factor $a^4+a^2b^2+b^4$.

By § 91, a trinomial is a perfect square if its first and last terms are perfect squares and positive, and its second term plus or minus twice the product of their square roots.

The given expression can be made a perfect square by adding a^2b^2 to its second term; and this can be done provided we subtract a^2b^2 from the result.

Thus, $a^4+a^2b^2+b^4=(a^4+2a^2b^2+b^4)-a^2b^2$

$=(a^2+b^2)^2-a^2b^2$, by § 91,

$=(a^2+b^2+ab)(a^2+b^2-ab)$, by § 89,

$=(a^2+ab+b^2)(a^2-ab+b^2)$.

2. Factor $9x^4-37x^2+4$.

The expression will be a perfect square if its second term is $-12x^2$.

Thus, $9x^4-37x^2+4=(9x^4-12x^2+4)-25x^2$

$=(3x^2-2)^2-(5x)^2$

$=(3x^2+5x-2)(3x^2-5x-2)$.

The expression may also be factored as follows:

$9x^4-37x^2+4=(9x^4+12x^2+4)-49x^2$

$=(3x^2+2)^2-(7x)^2=(3x^2+7x+2)(3x^2-7x+2)$.

Several expressions in Exercise 36 may be factored in two different ways.

The factoring of trinomials of the form $x^4+ax^2y^2+y^4$, when the factors involve surds, will be considered in § 237.

EXERCISE 36

Factor the following:

1. x^4+5x^2+9.
2. $a^4-21a^2b^2+36b^4$.
3. $4-33x^2+4x^4$.
4. $25m^4-14m^2n^2+n^4$.
5. $9x^4+6x^2y^2+49y^4$.
6. $16a^4-81a^2+16$.
7. $64-64m^2+25m^4$.
8. $49a^4-127a^2x^2+81x^4$.

Factor each of the following in two different ways (compare §§ 92, 94):

9. x^4-17x^2+16.
10. $9-148a^2+64a^4$.
11. $16m^4-104m^2x^2+25x^4$.
12. $36a^4-97a^2m^2+36m^4$.

97. Type VI. We find by division,

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2. \qquad \frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

That is,

If the sum of the cubes of two numbers be divided by the sum of the numbers, the quotient is the square of the first number, minus the product of the first by the second, plus the square of the second number.

If the difference of the cubes of two numbers be divided by the difference of the numbers, the quotient is the square of the first number, plus the product of the first by the second, plus the square of the second number.

If an expression can be resolved into three equal factors, it is said to be a *perfect cube*, and one of the equal factors is called its *cube root*.

Thus, since $27a^6b^3$ is equal to $3a^2b\times 3a^2b\times 3a^2b$, it is a perfect cube, and $3a^2b$ is its cube root.

Similarly to extract the cube root of a positive monomial perfect cube:

Extract the cube root of the numerical coefficient, and divide the exponent of each letter by 3.

Thus, the cube root of $125\,a^6b^9c^3$ is $5\,a^2b^3c$.

1. Divide $1+8\,a^3$ by $1+2\,a$.

By § 88, $8\,a^3$ is the cube of $2\,a$; then, by the first rule,

$$\frac{1+8\,a^3}{1+2\,a}=\frac{1+(2\,a)^3}{1+2\,a}=1-2\,a+(2\,a)^2=1-2\,a+4\,a^2.$$

(Compare Exs. 11–14, 26, Exercise 23.)

2. Divide $27\,x^6-64\,y^9$ by $3\,x^2-4\,y^3$.

By the second rule,

$$\frac{27\,x^6-64\,y^9}{3\,x^2-4\,y^3}=\frac{(3\,x^2)^3-(4\,y^3)^3}{3\,x^2-4\,y^3}=(3\,x^2)^2+(3\,x^2)(4\,y^3)+(4\,y^3)^2$$
$$=9\,x^4+12\,x^2y^3+16\,y^6.$$

EXERCISE 37

Find, without actual division, the values of the following:

1. $\dfrac{x^3+1}{x+1}$.
2. $\dfrac{1-a^3}{1-a}$.
3. $\dfrac{n^3-27}{n-3}$.
4. $\dfrac{a^6+b^6}{a^2+b^2}$.
5. $\dfrac{a^3+125}{a+5}$.
6. $\dfrac{64\,x^{6m}-1}{4\,x^{2m}-1}$.
7. $\dfrac{27\,x^6-125\,y^3}{3\,x^2-5\,y}$.
8. $\dfrac{343\,m^3n^3+8\,p^3}{7\,mn+2\,p}$.
9. $\dfrac{64\,a^6b^3+216\,c^9}{4\,a^2b+6\,c^3}$.

Factor the following:

10. a^3+b^3.
11. x^3-y^3.
12. $1+m^3n^3$.
13. $8\,a^3+27\,c^3$.
14. $1-27\,n^3$.
15. a^6+1.
16. $64\,m^3-n^3$.
17. $a^3b^3-216\,c^3$.
18. $8\,m^{3p}+27\,n^{3q}$.

98. Type VII. We find by actual division,

$$\frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3.$$
$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3.$$
$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4.$$
$$\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4;\ \text{etc.}$$

In these results, we observe the following laws:

I. The exponent of a in the first term of the quotient is less by 1 than its exponent in the dividend, and decreases by 1 in each succeeding term.

II. The exponent of b in the second term of the quotient is 1, and increases by 1 in each succeeding term.

III. If the divisor is $a-b$, all the terms of the quotient are positive; if the divisor is $a+b$, the terms of the quotient are alternately positive and negative.

(Compare Exs. 14, 16, 17, Exercise 23.)

1. Divide a^7-b^7 by $a-b$.

By the above laws,

$$\frac{a^7-b^7}{a-b}=a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6.$$

2. Divide $16\,x^4-81$ by $2\,x+3$.

We have
$$\frac{16\,x^4-81}{2\,x+3}=\frac{(2\,x)^4-3^4}{2\,x+3}$$
$$=(2\,x)^3-(2\,x)^2\cdot 3+2\,x\cdot 3^2-3^3$$
$$=8\,x^3-12\,x^2+18\,x-27.$$

EXERCISE 38

Find, without actual division, the values of the following:

1. $\dfrac{h^4-k^4}{h-k}$.
2. $\dfrac{m^5+n^5}{m+n}$.
3. $\dfrac{x^6-1}{x-1}$.
4. $\dfrac{1-a^6}{1+a}$.
5. $\dfrac{x^8-y^8}{x^2+y^2}$.
6. $\dfrac{a^{15}-b^5c^{10}}{a^3-bc^2}$.

Factor the following:

7. x^5+y^5.
8. a^5-1.
9. $1-m^5n^5$.
10. $1+x^7$.
11. a^7-b^7.
12. a^9-1.
13. x^9+n^9.
14. $32\,a^5-b^5$.
15. $n^{5p}+32$.
16. $243\,x^5+y^5$.
17. $m^{14}+128\,n^7$.
18. $32\,a^5b^{15n}-243\,c^{10p}$.

99. The following statements will be found helpful if n is a positive integer:

$x+y$ is a factor of x^n+y^n if n is *odd*.
$x-y$ is *never* a factor of x^n+y^n.
$x-y$ is *always* a factor of x^n-y^n.
$x+y$ is a factor of x^n-y^n if n is *even*.

When one factor is $x-y$ all the terms of the other factor are *positive*, and when one factor is $x+y$ the terms of the other factor are *alternately positive* and *negative*.

100. A Common Factor of two or more expressions is an expression which is a factor of each of them.

101. Type VIII. When the terms of the expression have a common factor.

1. Factor $14\,ab^4-35\,a^3b^2$.

Each term contains the monomial factor $7\,ab^2$.
Dividing the expression by $7\,ab^2$, we have $2\,b^2-5\,a^2$.
Then, $14\,ab^4-35\,a^3b^2=7\,ab^2(2\,b^2-5\,a^2)$.

2. Factor $(2\,m+3)x^2+(2\,m+3)y^3$.

The terms have the common binomial factor $2\,m+3$.
Dividing the expression by $2\,m+3$, we have x^2+y^3.

$$2\,m+3\overline{)(2\,m+3)x^2+(2\,m+3)y^3}\quad x^2+y^3.$$

Then, $(2\,m+3)x^2+(2\,m+3)y^3=(2\,m+3)(x^2+y^3)$.
(See example 6, Exercise 22.)

3. Factor $(a-b)m+(b-a)n$.

By § 46, $b-a=-(a-b)$.
Then, $(a-b)m+(b-a)n=(a-b)m-(a-b)n$
$=(a-b)(m-n)$.

We may also solve Ex. 3 as follows:

$$(a-b)m+(b-a)n=(b-a)n-(b-a)m=(b-a)(n-m).$$

4. Factor $5\,a(x-y)-3\,a(x+y)$.

$5\,a(x-y)-3\,a(x+y)=a[5(x-y)-3(x+y)]$
$=a(5\,x-5\,y-3\,x-3\,y)$
$=a(2\,x-8\,y)=2\,a(x-4\,y)$.

After a common factor is removed one or both of the factors may admit of further factoring.

5. Factor $a^2x^2+2\,a^2xy+a^2y^2$.

Dividing by the common factor a^2, we have for the factors a^2 and $x^2+2\,xy+y^2$.

The trinomial is factorable by § 91.

Whence, $a^2x^2+2\,a^2xy+a^2y^2=a^2(x+y)(x+y)$
$=a^2(x+y)^2$.

EXERCISE 39

Factor the following:

1. $36\,m^3-48\,m^2k^2$.
2. $a^5-3\,a^4b+3\,a^3b^2-a^2b^3$.
3. $21\,x^2y-33\,xy^2+12\,xy$.
4. $14\,z^2xc-28\,z^3x^2+7\,z^4xc^2$.
5. $(a+2)c^4-(a+2)d^3a$.
6. $(2\,x+7)x^3+(2\,x+7)$.
7. $(h-k)a^2-(k-h)4\,c^2$.
8. $c^2(c^2-2)+4\,y^2(2-c^2)$.
9. $(x+y)^2+4\,k(x+y)$.
10. $4\,d^3(d-1)-(1-d)$.
11. $4(3\,x+2)+4(2\,x+3)$.
12. $(a-x)^3-5(a-x)^2$.
13. $(2\,m+3)a^2-(2\,m+3)b^2$.
14. $(m-1)a^4-(m-1)b^4$.
15. $(a+b)a^2+(a+b)2\,ab+(a+b)b^2$.
16. $x^2(x^2-1)-8\,x(x^2-1)+15(x^2-1)$.
17. $(m-d)^4-2\,m(m-d)^3+m^2(m-d)^2$.

In every expression to be factored first remove the common factor, if any, then factor the remaining part if possible.

Sometimes it is necessary to group the terms (§§ 46, 47), to show a common factor, then apply the method of Type VIII.

18. $ab-ay+bx-xy$.

By § 46, $ab-ay+bx-xy=a(b-y)+x(b-y)$.

The terms now have the common factor $b-y$.

Whence, $ab-ay+bx-xy=(b-y)(a+x)$.

19. $a^3+2\,a^2-3\,a-6$.

If the third term is negative it is convenient to write the last two terms in parenthesis preceded by a $-$ sign, § 46.

Thus, $a^3+2\,a^2-3\,a-6=(a^3+2\,a^2)-(3\,a+6)$
$=a^2(a+2)-3(a+2)$
$=(a+2)(a^2-3)$.

20. $ac+ad+bc+bd$.
21. $xy-3x+2y-6$.
22. $mx+my-nx-ny$.
23. $ab-a-5b+5$.
24. $8xy+12ay+10bx+15ab$.
25. $m^4+6m^3-7m-42$.
26. $6-10a+27a^2-45a^3$.
27. $20ab-28ad-5bc+7cd$.

Be sure that the factors of your final result will not admit of further factoring.

28. $x^3+2x^2y+xy^2$.
29. $a+6ab+9ab^2$.
30. $x^2(a+b)-49y^2(a+b)$.
31. $108k^2s^3-36ks^3+3s^3$.
32. $m^2(2m+3)-3m(2m+3)-10(2m+3)$.
33. $9t^2(3t+2)+8t(3t+2)+4(3t+2)$.
34. $d^3(d+3c)+27c^3(d+3c)$.
35. $5a^2x^2-10a^2xy+5a^2y^2-20a^2z^2$.
36. $48a^6-243a^2b^4$.

Solve the following by inspection:

37. $98^2=(100-2)^2$
$=(10000-400+4)$
$=9604$.

38. $99^2=?$
39. $104^2=?$
40. $35^2=?$
41. $65^2=?$
42. $98^2-2^2=?$
43. $102^2-98^2=?$
44. $68^2=?$
45. $78^2=?$
46. $76^2-4^2=?$
47. $97^2-93^2=?$
48. $111^2-11^2=?$

The examples under Type IV afford a valuable application of the method in Type VIII.

49. Factor $6x^2-7x-20$.

Multiply 20 by 6 (the coefficient of x^2). Factor -120 so that the sum of the factors is -7 (the coefficient of x). These factors are -15, 8.

Then write $6x^2-7x-20=6x^2-15x+8x-20$.

Group by Type VIII, $=3x(2x-5)+4(2x-5)$,

whence, $6x^2-7x-20=(2x-5)(3x+4)$.

50. Factor examples 1–10, Exercise 34, by this method.

102. Hints on Factoring.

For *all expressions:*

First, try Type VIII.

Sometimes the common factor is disguised as in examples 8 and 19, Exercise 39.

Second, select the *type form* to which the expression belongs:

Test *binomials* by means of Types I, VI, VII.

Sometimes the binomial form is disguised. See examples 26 and 28, Exercise 40.

Test *trinomials* by means of Types II, III, IV, V.

Third, be sure that no factor in the result will admit of further factoring.

TYPE FORMS

I. $a^2-b^2=(a+b)(a-b)$. (§ 89)

II. $a^2+2ab+b^2=(a+b)(a+b)$,
$a^2-2ab+b^2=(a-b)(a-b)$. (§ 91)

III. x^2+ax+b. (§ 92)

IV. ax^2+bx+c. (§ 94)

V. $x^4+ax^2y^2+y^4$. (§ 96)

VI. $a^3+b^3=(a+b)(a^2-ab+b^2)$,
$a^3-b^3=(a-b)(a^2+ab+b^2)$. (§ 97)

VII. a^n-b^n,
a^n+b^n. (§ 98)

VIII. $ax+ay+az=a(x+y+z)$. (§ 101)

MISCELLANEOUS AND REVIEW EXAMPLES

EXERCISE 40

Factor the following:

1. $42\,a^2bc-7\,ab$.

2. $x^2-5\,x-36$.

3. $3\,a(a-x)+3\,a(c+d)$.

4. $36\,d^2-72\,dR+35\,R^2$.

5. a^4-64.

6. $v+\frac{1}{273}vt$.

7. $8a^2-14ab-15b^2$.

8. $27x^3-8z^3$.

9. $vt+\frac{1}{2}gt^2$.

10. c^4-d^4.

11. $125a^4-50a^3b+5a^2b^2$.

12. $a(b+c)-a(b-c)$.

13. $x^2(5y-2z)-x^2(2y+z)$.

14. $a^6-16a^3c^3+64c^6$.

15. a^6-26a^3-27.

16. $x^{14}-2x^7+1$.

17. $ax-ay+az-bx+by-bz$.

18. $(a+b)^2+14(a+b)+24$.

19. $(x-y)^2-15(x-y)-16$.

20. $4c^2(c+d)+12cd(c+d)+9d^2(c+d)$.

21. $2c^2(2c+3d)+5cd(2c+3d)+3d^2(2c+3d)$.

22. $18x^2-27abx-35a^2b^2$.

23. $x^2+(5c+2d)x+10cd$.

24. $7x^2(3a-2b)-3x^2(2a-3b)$.

After factors are found always unite any similar terms which occur in parenthesis.

25. $(x^2+x-2)^2-(x^2-x+3)^2$.

26. $64a^3+1000$.

27. $a^2-c^2-d^2+b^2-2ab-2cd$.

28. $36m^2-(x-y)^2+12m+1$.

29. $3(m+n)^2-2(m^2-n^2)$.

30. $2a^7x-8a^5x^3+2a^3x^5-8ax^7$.

31. $2xy-2x^2y^2-264x^3y^3$.

32. $8a(2-3y+x)+5c(3y-x-2)$.

33. h^2-k^2+h+k.

34. m^3+m+x^3+x.

Find the factors common (§ 100) to the following expressions:

35. x^2+x-6, $4x^2-11x+6$.

36. a^2-9c^2, $a^2+4ac-21c^2$, a^3-27c^3.

37. $xy+3cx+2cy+6c^2$, $y^3-5cy^2-24c^2y$.

38. $x(x^2+2x+2)+2(x^2+2x+2)$, x^4+4.

39. $(2c-y)4c^2-(2c-y)4cy+(2c-y)y^2$, $(2c-y)^3$

40. $6a^2+a-2$, $90a^3-25a^2-10a$, $4a^2+2a-2$.

41. x^3+2x^2+2x+1, x^3+1.

42. Solve, using factoring: A square, 441 feet on a side, has a grass plot within it, 432 feet on a side. The remaining part of the square is a concrete walk. Find the cost of the walk at 14¢ per square foot.

Additional work in factoring will be found in §§ 236 and 237.

SOLUTION OF EQUATIONS BY FACTORING

103. The solution of equations affords an important and interesting application of factoring.

Let it be required to solve the equation

$$(x-3)(2x+5)=0.$$

It is evident that the equation will be satisfied when x has such a value that *one* of the factors of the first member is equal to zero; for if any factor of a product is equal to zero, the product is equal to zero.

Hence, the equation will be satisfied when x has such a value that either

$$x-3=0, \tag{1}$$

or

$$2x+5=0. \tag{2}$$

Solving (1) and (2), we have $x=3$ or $-\frac{5}{2}$.

It will be observed that the roots are obtained by placing the factors of the first member separately equal to zero, and solving the resulting equations.

104. Examples.

1. Solve the equation $x^2-5x-24=0$.

Factoring the first member, $(x-8)(x+3)=0$. (§ 92)

Placing the factors separately equal to 0 (§ 103), we have

$$x-8=0, \text{ whence } x=8;$$

and

$$x+3=0, \text{ whence } x=-3.$$

Verify by substituting $x=8$, $x=-3$ successively in the given equation.

2. Solve the equation $4x^2-2x=0$.

Factoring the first member, $2x(2x-1)=0$.

Placing the factors separately equal to 0, we have

$$2x=0, \text{ whence } x=0;$$

and $$2x-1=0, \text{ whence } x=\frac{1}{2}.$$

Verify these results.

3. Solve the equation $x^3+4x^2-x-4=0$.

Factoring the first member, we have by §§ 89, 101,

$$(x+4)(x^2-1)=0, \text{ or } (x+4)(x+1)(x-1)=0.$$

Then, $$x+4=0, \text{ whence } x=-4;$$

$$x+1=0, \text{ whence } x=-1;$$

and $$x-1=0, \text{ whence } x=1.$$

Verify these results.

4. Solve the equation $x^3-27-(x^2+9x-36)=0$.

Factoring the first member, we have by §§ 92 and 97,

$$(x-3)(x^2+3x+9)-(x-3)(x+12)=0.$$

Or, $$(x-3)(x^2+3x+9-x-12)=0.$$

Or, $$(x-3)(x^2+2x-3)=0.$$

Or, $$(x-3)(x+3)(x-1)=0.$$

Placing the factors separately equal to 0, $x=3$, -3, or 1. Verify.

The pupil should endeavor to put down the values of x without actually placing the factors equal to 0, as shown in Ex. 4.

EXERCISE 41

Solve each equation and verify results:

1. $x^2-4x-21=0$.
2. $x^2-4x=0$.
3. $6x^3-12x^2=0$.
4. $(2x-7)(x^2-16)=0$.
5. $t^2-t-12=0$.
6. $z^2-8z+12=0$.
7. $k^2+7k+12=0$.
8. $6v^2-17v+12=0$.
9. $9v^2(2v-3)-9v(2v-3)-4(2v-3)=0$.
10. $3x^2-kx-4k^2=0$. (Solve for x, then solve for k.)
11. $10u^2-7u-12=0$.
12. $xz+2x-3z-6=0$.
13. $15v^2+v-2=0$
14. $4x^3+20x^2-9x-45=0$.
15. $28t^2-t-2=0$.
16. $18x^2-27abx-35a^2b^2=0$.

17. $n^2+14\,n-32=0$.

18. $x^2+8\,x+16=0$.

19. $m^3+6\,m^2-9\,m-54=0$.

20. $(x-3)^2-(3\,x+2)^2=0$.

21. $10\,v^2-39\,v+14=0$.

22. $15\,x^2+x-6=0$.

23. $(4\,x^2-49)(x^2-3\,x-10)(8\,x^2+14\,x-15)=0$.

24. $(x-2)(5\,x^2+8\,x-4)-(x^2-4)=0$.

25. What number added to its square gives 30?

26. What number subtracted from 4 times its square gives $\frac{1}{2}$?

27. If to 4 times the square of a certain number we add three times the number the result is 10. Find the number.

28. A rectangular room is 4 feet longer than it is wide, and its area is 96 square feet. What are its dimensions?

Let $w=$ the number of feet in the width,
then $w+4=$ the number of feet in the length.

$$w(w+4)=96,$$
$$w^2+4\,w-96=0,$$
$$(w-8)(w+12)=0.$$

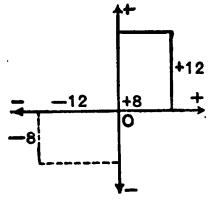
Whence, $w=8$ or -12.

Then, $w+4=12$ or -8.

Since we are finding dimensions of a room, these negative roots have no significance and can be rejected. There is, however, a very interesting geometrical interpretation which may be given. Consider § 10 and Exercise 4. If measurement to the right is positive, then measurement to the left is negative. If distance upward is +, then distance downward is −.

Now draw this rectangle:

This gives two rectangles which fulfill the conditions of the problem, if one remembers that -12 is algebraically less than -8.



29. In a right triangle ABC, the base, AC, is 3 feet more than the altitude, BC, and the area is 14 square feet. Find AC and BC. Make a diagram with your results.

30. The perimeter of a rectangular field is 180 feet, and its area 1800 square feet. Find its dimensions. Make a diagram of your results.

Find the equations whose roots (§ 73) are:

31. $2, -\frac{5}{3}$.

Subtracting each root from x, we have

$$(x-2), \quad (x-\tfrac{5}{3}).$$

By reversing § 103, the product of these expressions equated to zero gives the required equation.

Whence, $(x-2)(x+\frac{5}{3})=0$, or expanding,

$$3x^2-x-10=0.$$

32. 1, 3.

33. $\frac{2}{5}, \frac{5}{2}$.

34. $-1, 4$.

35. $2, -3, 4$.

36. a, b.

37. $\frac{a}{2}, \frac{a}{4}, a$.

38. $6, -1\frac{3}{2}$.

39. $1\frac{6}{9}, 0$.

40. $a+2b, a-2b$.

41. The sides of a rectangle are 8 and 11. Form a problem similar to problem 28. State the equation.

QUERIES

1. Is $2a$ a number? Is it a sum? Is it a product? What are its factors?
2. Is $a+b$ a number? Is it the sum of two numbers? Can you factor it?
3. Translate a^2+b^2 into English. Can you factor it?
4. Given two numbers F and S; if their sum is multiplied by their difference, what is the result?
5. Given two numbers F and S; if their sum be multiplied by itself, what is the result? Express in English.
6. Does the *definition of division* bear any relation to your idea of the *process of factoring?*
7. Is $4a^2+2a+1$ a perfect square? Why?
8. The following are for mental drill: $(30\frac{1}{2})^2=?$ $(20\frac{1}{4})^2=?$ $(29\frac{1}{2})^2=?$
9. Is 3 a root of the equation $3x^2-4x+7=0$? Why? Is $x-3$ a factor of the expression?
10. Is 2 a root of $2m^2-9m+10=0$? Is $m-2$ a factor of the expression?
11. How do you form the equation whose roots are 3 and 7?
12. If one root, 5, of $x^2-8x+15=0$ is given, can you find the other root without solving the equation?
13. Using your knowledge of § 91, can you make a general statement covering the results of examples 13 and 14, Exercise 12?

VIII. HIGHEST COMMON FACTOR. LOWEST COMMON MULTIPLE

(We consider in the present chapter the Highest Common Factor and Lowest Common Multiple of *Monomials*, or of *Polynomials* which can be readily factored by inspection.

The Highest Common Factor and Lowest Common Multiple of polynomials which cannot be readily factored by inspection, will be considered in a more advanced course in algebra.)

HIGHEST COMMON FACTOR

105. The Highest Common Factor (H. C. F.) **of two or more expressions is their common factor of *highest degree*** (§ 58).

If several common factors are of equally high degree, it is understood that *the* highest common factor is the one having the numerical coefficient of greatest absolute value in its term of highest degree.

For example, if the common factors were $6x$ and $2x$, the former would be the H. C. F.

106. Two expressions are said to be *prime to each other* when unity is their highest common factor.

107. Case I. Highest Common Factor of Monomials.

Ex. Required the H. C. F. of $42a^3b^2$, $70a^2bc$, and $98a^4b^3d^2$.

By the rule of Arithmetic, the H. C. F. of 42, 70, and 98 is 14.

It is evident by inspection that the expression of highest degree which will exactly divide a^3b^2, a^2bc, and $a^4b^3d^2$ is a^2b.

Then, the H. C. F. of the given expressions is $14a^2b$.

It will be observed, in the above result, that *the exponent of each letter is the lowest exponent with which it occurs in any of the given expressions.*

EXERCISE 42

Find the H. C. F. of the following:

1. $14x^3y$, $21xy^4$.
2. $64a^5b^3$, $112b^4c^4$.
3. $36m^3b^2$, $48m^2b^3$, $60m^4b$.
4. $25ac^2$, $30a^2c^2$, $35ac$.

5. $32\,a^4x^4$, $128\,a^6b^2x^3$, $192\,a^3x^2y^3$.

6. $136\,a^4m^5n^3$, $51\,b^3mn^6$, $119\,c^2m^3n^9$.

7. $60(x-y)^4$, $84(x-y)^5$.

108. By § 48, $(+a)\times(+b)=+ab$, $(+a)\times(-b)=-ab$,
$(-a)\times(+b)=-ab$, $(-a)\times(-b)=+ab$.

Hence, in the indicated product of two factors, the signs of both factors may be changed without altering the product; but if the sign of either one be changed, the sign of the product will be changed.

If either factor is a polynomial, care must be taken, on changing its sign, to change the sign of *each of its terms*.

Thus, $(b-a)(n-m)$ may be written in the form
$-(b-a)(m-n)$, or $-(a-b)(n-m)$.

In like manner, in the indicated product of more than two factors, the signs of any *even* number of them may be changed without altering the product; but if the signs of any *odd* number of them be changed, the sign of the product will be changed (§ 49).

Thus, $(a-b)(c-d)(e-f)$ may be written in the forms
$(a-b)(d-c)(f-e)$,
$(b-a)(c-d)(f-e)$,
$-(b-a)(d-c)(f-e)$, etc.

109. Case II. Highest Common Factor of Polynomials which can be readily factored by Inspection.

1. Required the H. C. F. of
$5\,x^4y-45\,x^2y$ and $10\,x^3y^2-40\,x^2y^2-210\,xy^2$.

By §§ 101, 89, and 92, $5\,x^4y-45\,x^2y=5\,x^2y(x^2-9)$
$=5\,x^2y(x+3)(x-3)$; (1)
and $10\,x^3y^2-40\,x^2y^2-210\,xy^2=10\,xy^2(x^2-4\,x-21)$
$=10\,xy^2(x-7)(x+3)$. (2)

The H. C. F. of the numerical coefficients 5 and 10 is 5.

It is evident by inspection that the H. C. F. of the literal portions of the expressions (1) and (2) is $xy(x+3)$.

Then, the H. C. F. of the given expressions is $5\,xy(x+3)$.

It is sometimes necessary to change the form of the factors in finding the H. C. F. of expressions.

2. Find the H. C. F. of a^2+2a-3 and $1-a^3$.

By § 92, $a^2+2a-3=(a-1)(a+3)$.

By § 97, $1-a^3=(1-a)(1+a+a^2)$.

By § 108, the factors of the first expression can be put in the form

$$-(1-a)(3+a).$$

Hence, the H. C. F. is $1-a$.

EXERCISE 43

Find the H. C. F. of the following:

1. $10x^3y^2-40x^2y^3$, $25xy^2-15xy^4$.
2. c^2-25b^2, $c^2-10bc+25b^2$.
3. $a^2-5a-36$, $a^2-4a-32$.
4. $tz+5z-7t-35$, $t^2z+8tz+15z$.
5. $2a^2-ab-3b^2$, $3a^2+ab-2b^2$.
6. $9a^2-25b^2$, $9a^2-30ab+25b^2$.
7. $8n^3+1$, $4n^2-2n+1$.
8. $n^2-3n-40$, n^2+4n-5, $2n^2+6n-20$.
9. t^3+2t^2+t+2, t^4+3t^2+2.
10. v^3-v^2-v+1, v^4-2v^2+1.
11. $6a^2+a-2$, $12a^2+5a-2$, $6a^2+4a-15az-10z$.
12. $25k^2-16$, $25k^2-40k+16$, $30k^2+k-20$.
13. a^5-32, $a^2+9a-22$, a^3-8.
14. $1-11a+18a^2$, $8a^3-1$, $18a^2-5a-2$.
15. x^4+3x^2-40, x^4-25, a^3+a^2-5a-5.
16. $a^2-(b+c)^2$, $(b-a)^2-c^2$, $b^2-(a-c)^2$.
17. $(x^2+x+2)(x^2-x-2)$, x^2-5x-6, x^2-8x-9.
18. $2a^2(2a+3t)+5at(2a+3t)+3t^2(2a+3t)$ and $4a^2(t+a)+12at(t+a)+9t^2(t+a)$.

LOWEST COMMON MULTIPLE

110. A Common Multiple of two or more expressions is an expression which is exactly divisible by each of them.

111. The Lowest Common Multiple (L. C. M.) of two or more expressions is their common multiple of *lowest degree*.

If several common multiples are of equally low degree, it is understood that *the* lowest common multiple is the one having the numerical coefficient of least absolute value in its term of highest degree.

For example, if the common multiples were $4x-2$ and $6x-3$, the former would be the L. C. M.

112. Case I. Lowest Common Multiple of Monomials.

Ex. Required the L. C. M. of $36a^3x$, $60a^2y^2$, and $84cx^3$.

By the rule of Arithmetic, the L. C. M. of 36, 60, and 84 is 1260.

It is evident by inspection that the expression of lowest degree which is exactly divisible by a^3x, a^2y^2, and cx^3 is $a^3cx^3y^2$.

Then, the L. C. M. of the given expressions is $1260a^3cx^3y^2$.

It will be observed, in the above result, that *the exponent of each letter is the highest exponent with which it occurs in any of the given expressions.*

EXERCISE 44

Find the L. C. M. of the following:

1. $5x^3y^3$, $6x^2y^4$.
2. $18a^6b$, $45b^5c$.
3. $28x^5$, $36y^4$.
4. $42m^4n^2$, $98n^5p^3$.
5. $105a^2b$, $70b^3c$, $63c^3a$.
6. $50x^4y^5$, $24x^5y^3$, $40x^6y^4$.
7. $21ab^4$, $35b^2c^6$, $91a^3c^3$.
8. $56a^2b^3$, $84bx^5$, $48x^4y^6$.
9. $60a^3bc^2$, $75a^5b^6d$, $90a^4c^7d^6$.
10. $99m^4nx^8$, $66m^3n^9y^5$, $165n^5x^4y^7$.

113. Case II. Lowest Common Multiple of Polynomials which can be readily factored by Inspection.

1. Required the L. C. M. of

$$x^2-5x+6,\ x^2-4x+4,\ \text{and}\ x^3-9x.$$

By § 92, $x^2-5x+6=(x-3)(x-2)$.

By § 91, $x^2-4x+4=(x-2)^2$.

By § 89, $x^3-9x=x(x+3)(x-3)$.

It is evident by inspection that the L. C. M. of these expressions is

$$x(x-2)^2(x+3)(x-3).$$

It is sometimes necessary to change the form of the factors.

2. Find the L. C. M. of $ac-bc-ad+bd$ and b^2-a^2.

By § 101, $ac-bc-ad+bd=(a-b)(c-d)$.

By § 89, $b^2-a^2=(b+a)(b-a)$.

By § 108, the factors of the first expression can be written

$$(b-a)(d-c).$$

Hence, the L. C. M. is $(b+a)(b-a)(d-c)$, or $(b^2-a^2)(d-c)$.

EXERCISE 45

Find the L. C. M. of the following:

1. $x^2-7x+10$, $x^2-8x+15$.
2. k^3-4, $k^3-7k+10$, $k^3-5k^2+4k-20$.
3. $2a^2-a-1$, $2a^2+5a+2$, $2a^2+7a+3$.
4. R^2-3R+2, R^2-5R+6, R^2-4R+3.
5. a^2-2a-3, a^2-3a+2, a^2-1.
6. $m-2$, m^2-2m+4, $m^3-6m^2+12m-8$.
7. x^4+x^2+1, x^2+x+1, x^2-x+1.
8. $k+l$, $k+3l$, $l-k$, $k-3l$.
9. $(x-2)(x-3)$, $(x-3)(x-4)$, $(4-x)(2-x)$.
10. a^2-9, a^3-27, $a-3$, a^2+3a+9.

Find the H. C. F. and the L. C. M. of the following:

11. m^2+3m+2, m^2+5m+6, m^2+4x+3.
12. $a^2+4ab+4b^2$, a^2-4b^2, a^2+2ab.
13. $2k^2+7k-4$, $3k^2+13k+4$.
14. $2x^2-3ax+a^2$, $2x^2-5ax+2a^2$, $x^2-3ax+2a^2$.
15. $9t^2-25v^2$, $6tx+10vx$, $12tx+20vx$.

Find the L. C. M. of the following:

16. $9x^3-12x^2+4x$, $18ax^4+12ax^3+8ax^2$, and $27x^3-8$.
17. $(x+z)^2-y^2$, $(x+y)^2-z^2$, $x^2-(y+z)^2$.
18. $(c-1)^2+3c$, c^3-1, $c-1$.
19. $(e+y)^2-4ey$, $e^3+2e^2y+ey^2$, e^4+ey^3.
20. x^3+8, $4x^2-(x^2+4)^2$.

IX. FRACTIONS

114. The quotient of a divided by b is written $\frac{a}{b}$.

The expression $\frac{a}{b}$ is called a **Fraction**; the dividend a is called the *numerator*, and the divisor b the *denominator*.

The numerator and denominator are called the *terms* of the fraction.

115. It follows from § 62, (3), that

If the terms of a fraction be both multiplied, or both divided, by the same expression, the value of the fraction is not changed.

116. By the Rule of Signs in Division (§ 61),

$$\frac{+a}{+b} = \frac{-a}{-b} = -\frac{+a}{-b} = -\frac{-a}{+b}.$$

That is, if the signs of both terms of a fraction be changed, the sign before the fraction is not changed; but if the sign of either one be changed, the sign before the fraction is changed.

If either term is a polynomial, care must be taken, on changing its sign, to change the sign of *each of its terms*.

Thus, the fraction $\frac{a-b}{c-d}$, by changing the signs of both numerator and denominator, can be written $\frac{b-a}{d-c}$ (§ 44).

117. It follows from §§ 108 and 116 that **if either term of a fraction is the indicated product of two or more factors, the signs of any *even* number of them may be changed without changing the sign before the fraction: but if the signs of any *odd* number of them be changed, the sign before the fraction is changed.**

Note: To change the sign of a factor is to change the sign of every term of the factor.

Thus, the fraction $\frac{a-b}{(c-d)(e-f)}$ may be written

$$\frac{a-b}{(d-c)(f-e)}, \frac{b-a}{(d-c)(e-f)}, -\frac{b-a}{(d-c)(f-e)}, \text{ etc.}$$

EXERCISE 46

Write each of the following in three other ways without changing its value:

1. $\frac{a}{2}$. 2. $\frac{n+3}{7}$. 3. $\frac{8}{2-x}$. 4. $\frac{2x-7}{x+2}$. 5. $\frac{6x-5}{(x-3)(y+4)}$.

6. Write $\frac{(3x-1)(a-4)}{(x+5)(y-2)}$ in four other ways without changing its value.

REDUCTION OF FRACTIONS

118. Reduction of a Fraction to Lower Terms.

A fraction is said to be in its *lowest terms* when its numerator and denominator are prime to each other (§ 106).

(We consider in this text those cases only in which the numerator and denominator can be readily factored by inspection.

The cases in which the numerator and denominator cannot be readily factored by inspection are considered in the second course.)

119. By § 115, dividing both terms of a fraction by the same expression, or cancelling common factors in the numerator and denominator, does not alter the value of the fraction.

We then have the following rule:

Resolve both numerator and denominator into their factors, and cancel all that are common to both.

1. Reduce $\frac{24\,a^4b^2cx}{40\,a^2b^2c^2d^3}$ to its lowest terms.

We have, $\frac{24\,a^4b^2cx}{40\,a^2b^2c^2d^3} = \frac{2^3\times 3\times a^4b^2cx}{2^3\times 5\times a^2b^2c^2d^3} = \frac{3\,a^2x}{5\,cd^3}$,

by cancelling the common factor $2^3\times a^2b^2c$.

2. Reduce $\frac{x^3-27}{x^2-2x-3}$ to its lowest terms.

By §§ 97 and 92, $\frac{x^3-27}{x^2-2x-3} = \frac{(x-3)(x^2+3x+9)}{(x-3)(x+1)} = \frac{x^2+3x+9}{x+1}$.

3. Reduce $\frac{ax-bx-ay+by}{b^2-a^2}$ to its lowest terms.

By §§ 89 and 101, $\dfrac{ax-bx-ay+by}{b^2-a^2}=\dfrac{(a-b)(x-y)}{(b+a)(b-a)}$.

By § 117, the signs of the terms of the factors of the numerator can be changed without altering the value of the fraction; and in this way the first factor of the numerator becomes the same as the second factor of the denominator.

Then, $\dfrac{ax-bx-ay+by}{b^2-a^2}=\dfrac{(b-a)(y-x)}{(b+a)(b-a)}=\dfrac{y-x}{b+a}$.

If all the factors of the numerator are cancelled, 1 remains to form a numerator; if all the factors of the denominator are cancelled, it is a case of exact division.

EXERCISE 47

Reduce each of the following to its lowest terms:

1. $\dfrac{5\,x^4y^5z^2}{3\,xy^5z^6}$.
2. $\dfrac{12\,a^5b^4}{42\,b^2c^3}$.
3. $\dfrac{54\,mn^2}{99\,m^5n^3}$.
4. $\dfrac{63\,x^3y^4z^7}{84\,x^5y^4z^2}$.
5. $\dfrac{126\,a^6b^3c^5}{14\,a^6c^4}$.
6. $\dfrac{26\,m^3n^4p^2}{130\,m^4n^6p^7}$.
7. $\dfrac{90\,a^3m^7n^4}{36\,am^7n^3}$.
8. $\dfrac{88\,x^4y^6z^3}{66\,x^5yz^9}$.
9. $\dfrac{120\,a^7b^4c^{10}}{75\,ab^9c^2}$.
10. $\dfrac{15\,x^4y+10\,x^3y^2}{6\,x^3y^4+4\,x^2y^5}$.
11. $\dfrac{x^2-9\,x+18}{x^2+x-12}$.
12. $\dfrac{m^2-5\,m-84}{a^2m^2-a^2m-56\,a^2}$.
13. $\dfrac{6\,x^2-7\,xz-20\,z^2}{4\,x^2-25\,z^2}$.
14. $\dfrac{4\,a^2-16\,ad+15\,d^2}{4\,a^2-12\,ad+9\,d^2}$.
15. $\dfrac{a^2+a-12}{3\,a^2-13\,a+12}$.
16. $\dfrac{(x^2-49)(x^2-16x+63)}{(x^2-14\,x+49)(x^2-2\,x-63)}$.
17. $\dfrac{12\,a^4m^2+48\,a^4-10\,m^2b-40\,b}{36\,a^3-60\,a^4b+25\,b^2}$.
18. $\dfrac{36\,a^2+97\,ac+36\,c^2}{9\,a^2+13\,ac+4\,c^2}$.
19. $\dfrac{18\,a^2-3\,ac-10\,c^2}{36\,a^2-25\,c^2}$.
20. $\dfrac{27\,b^3-8\,a^3}{16\,a^2-32\,ab+12\,b^2}$.
21. $\dfrac{165\,t^2+2\,t-1}{15\,t^4+14\,t^3-t^2}$.

22. $\dfrac{(4a-d)^2-(2b-3c)^2}{(4a-2b)^2-(3c-d)^2}$. 23. $\dfrac{(x-y)^2-(2v-u^2)^2}{(x-2v)^2-(u^2-y)^2}$.

24. $\dfrac{4c^2(2c-3d)-6cd(2c-3d)+9d^2(2c-3d)}{10c^2+cd-24d^2}$.

120. Reduction of a Fraction to an Integral or Mixed Expression.

A Mixed Expression is a polynomial consisting of a rational and integral expression (§ 57), with one or more fractions.

Thus, $a+\frac{b}{c}$, and $\frac{x}{3}+\frac{2x-y}{x-y}$ are mixed expressions.

121. A fraction may be reduced to an integral or mixed expression by the operation of division, if the degree (§ 58) of the numerator is equal to, or greater than, that of the denominator.

1. Reduce $\dfrac{6x^2+15x-2}{3x}$ to a mixed expression.

By § 65, $\dfrac{6x^2+15x-2}{3x}=\dfrac{6x^2}{3x}+\dfrac{15x}{3x}-\dfrac{2}{3x}=2x+5-\dfrac{2}{3x}$.

2. Reduce $\dfrac{12x^3-8x^2+4x-5}{4x^2+3}$ to a mixed expression.

$$\begin{array}{r|l} 4x^2+3)12x^3-8x^2+4x-5 & (3x-2 \\ \underline{12x^3\qquad +9x\quad} & \\ -8x^2-5x-5 & \\ \underline{-8x^2\qquad -6} & \\ -5x+1 & \end{array}$$

Since the dividend is equal to the product of the divisor and quotient, plus the remainder, we have

$$12x^3-8x^2+4x-5=(4x^2+3)(3x-2)+(-5x+1).$$

Dividing both members by $4x^2+3$, we have

$$\frac{12x^3-8x^2+4x-5}{4x^2+3}=3x-2+\frac{-5x+1}{4x^2+3}.$$

Thus, a remainder of lower degree than the divisor may be written over the divisor in the form of a fraction, and the result added to the quotient.

If the first term of the numerator is negative, as in Ex. 2, it is usual to *change the sign of each term of the numerator*, changing the sign before the fraction (§ 116).

Thus, $\dfrac{12x^3-8x^2+4x-5}{4x^2+3}=3x-2-\dfrac{5x-1}{4x^2+3}$.

EXERCISE 48

Reduce each of the following to a mixed expression:

1. $\dfrac{25a^2-10a+11}{5a}$.

2. $\dfrac{16m^4+12m^3+8m^2-9}{3m^2}$.

3. $\dfrac{4x^4+1}{2x^2+1}$.

4. $\dfrac{x^2+6x+9}{x^3+3x+9}$.

5. $\dfrac{x^3+y^3}{x-y}$.

6. $\dfrac{8a^3-27c^3}{2a+3c}$.

7. $\dfrac{8a^4+12a^2b^2-15b^4}{2a^2+3b^2}$.

8. $\dfrac{a^5+32}{a-2}$.

9. $\dfrac{c^4+d^4}{c+d}$.

10. $\dfrac{24x^3+21x+19}{4x^2-2x+5}$.

11. $\dfrac{x^2-4x+8}{x^2+2x-7}$.

12. $\dfrac{3x^3+7x^2}{3x^2+x-9}$.

13. $\dfrac{8a^2-22ab-21b^2}{2a-7b}$.

14. $\dfrac{49x^2-96xy+27y^2}{7x^2+xy-18y^2}$.

122. It is evident from § 121 that a mixed expression may be result of division. Since the dividend is equal to the product of the divisor and quotient plus the remainder, to reduce a mixed expression to a fraction, **Multiply the integral part by the denominator of the fraction, add the numerator to this result and write the denominator under this sum.**

Note: If a minus sign precedes the fraction, change each sign in the numerator.

Ex. Reduce $2x-3-\frac{4x-5}{x+1}$ to a fractional form.

$$2x-3-\frac{4x-5}{x+1}=\frac{(2x-3)(x+1)-(4x-5)}{x+1}$$
$$=\frac{2x^2-x-3-4x+5}{x+1}=\frac{2x^2-5x+2}{x+1}.$$

EXERCISE 49

1. $c-3+\frac{4c-3}{2c}$.
2. $5a-1-\frac{6a^2-2}{5a}$.
3. $2n+11+\frac{3}{6n+2}$.
4. $\frac{3x+4y}{3x-4y}+1$.
5. $\frac{a^2+2ab+b^2}{4ab}-1$.
6. $\frac{(x+y)^2}{(x-y)^2}-1$.
7. $\frac{(3c-8d)^2}{9c^2-64d^2}-1$.
8. $x-4a-\frac{-4a^2}{x}$.
9. $x+5a-\frac{20a}{x-5a}$.
10. $2t-3u-\frac{8t^2+27u^2}{4t^2+6tu+9u^2}$.
11. $2a-5b-\frac{4a^2-25b^2}{2a-5b}$.
12. $x^2+2xy+y^2+\frac{16y^2}{x-2y}$.
13. $\frac{9a^2-4ab+b^2}{4a-2b}-(2a-3b)$.
14. $\frac{10a^2-29ac+10c^2}{3a-c}+6a+2c$.

123. Reduction of Fractions to their Lowest Common Denominator. — To reduce fractions to their Lowest Common Denominator (L. C. D.) is to express them as equivalent fractions, each having for a denominator the L. C. M. of the given denominators.

Let it be required to reduce $\frac{4cd}{3a^2b^3}$, $\frac{3m}{2ab^2}$, and $\frac{5n}{4a^3b}$ to their lowest common denominator.

The L. C. M. of $3a^2b^3$, $2ab^2$, and $4a^3b$ is $12a^3b^3$ (§ 112).

By § 115, if the terms of a fraction be both multiplied by the same expression, the value of the fraction is not changed.

Multiplying both terms of $\frac{4cd}{3a^2b^3}$ by $4a$, both terms of $\frac{3m}{2ab^2}$ by $6a^2b$, and both terms of $\frac{5n}{4a^3b}$ by $3b^2$, we have

$$\frac{16\,acd}{12\,a^2b^2},\ \frac{18\,a^2bm}{12\,a^2b^2},\ \text{and}\ \frac{15\,b^2n}{12\,a^2b^2}.$$

It will be seen that the terms of each fraction are multiplied by an expression, which is obtained by dividing the L. C. D. by the denominator of this fraction.

Whence the following rule:

Find the L. C. M. of the given denominators.

Multiply both terms of each fraction by the quotient obtained by dividing the L. C. D. by the denominator of this fraction.

Before applying the rule, each fraction should be reduced to its lowest terms.

124. *Ex.* Reduce $\frac{4\,a}{a^2-4}$ and $\frac{3\,a}{a^2-5\,a+6}$ to their lowest common denominator.

We have $a^2-4=(a+2)(a-2)$,

and $a^2-5\,a+6=(a-2)(a-3)$.

Then, the L. C. D. is $(a+2)(a-2)(a-3)$. (§ 113)

Dividing the L. C. D. by $(a+2)(a-2)$, the quotient is $a-3$; dividing it by $(a-2)(a-3)$, the quotient is $a+2$.

Then, by the rule, the required fractions are

$$\frac{4\,a(a-3)}{(a+2)(a-2)(a-3)}\ \text{and}\ \frac{3\,a(a+2)}{(a+2)(a-2)(a-3)}.$$

EXERCISE 50

Reduce the following to their lowest common denominator:

1. $\frac{7\,ab}{6},\ \frac{3\,bc}{10},\ \frac{2\,ca}{15}.$

2. $\frac{5}{2\,m^2n},\ \frac{4}{5\,m^3n^2},\ \frac{6}{7\,mn^2}.$

3. $\frac{3\,x+4\,z}{22\,xy^3},\ \frac{6\,x-5\,y}{33\,yz^2}$

4. $\frac{11\,c^4p}{12\,a^2b},\ \frac{9\,a^5m}{14\,b^4c},\ \frac{8\,b^3n}{21\,c^3a}.$

5. $\frac{4\,a^2}{4\,a^2-9},\ \frac{2}{6\,a^2-9\,a}.$

6. $\frac{1}{m-n},\ \frac{3\,mn}{2(m-n)^2},\ \frac{2\,m^2n^2}{3(m-n)^3}.$

7. $\frac{3\,n}{n^2-8},\ \frac{5}{n^2-4\,n+4}.$

8. $\frac{2}{a^3+3\,a^2+2\,a+6},\ \frac{3\,a}{a^3+27}.$

ADDITION AND SUBTRACTION OF FRACTIONS

125. By § 65, $\frac{b}{a}+\frac{c}{a}-\frac{d}{a}=\frac{b+c-d}{a}$.

We then have the following rule:

To add or subtract fractions, reduce them, if necessary, to equivalent fractions having the lowest common denominator.

Add or subtract the numerator of each resulting fraction, according as the sign before the fraction is + or −, and write the result over the lowest common denominator.

The final result should be reduced to its lowest terms.

126. Examples.

1. Simplify $\frac{4a+3}{4a^2b}+\frac{1-6b^2}{6ab^3}$.

The L. C. D. is $12a^2b^3$; multiplying the terms of the first fraction by $3b^2$, and the terms of the second by $2a$, we have

$$\frac{4a+3}{4a^2b}+\frac{1-6b^2}{6ab^3}=\frac{12ab^2+9b^2}{12a^2b^3}+\frac{2a-12ab^2}{12a^2b^3}$$

$$=\frac{12ab^2+9b^2+2a-12ab^2}{12a^2b^3}=\frac{9b^2+2a}{12a^2b^3}.$$

If a fraction whose numerator is a polynomial is preceded by a − sign, it is convenient to write the numerator in parenthesis preceded by a − sign, as shown in the last term of the numerator in equation (A), of Ex. 2.

If this is not done, care must be taken to *change the sign of each term of the numerator* before combining it with the other numerators.

2. Simplify $\frac{5x-4y}{6}-\frac{7x-2y}{14}$.

The L. C. D. is 42; whence,

$$\frac{5x-4y}{6}-\frac{7x-2y}{14}=\frac{35x-28y}{42}-\frac{21x-6y}{42}$$

$$=\frac{35x-28y-(21x-6y)}{42} \qquad (A)$$

$$=\frac{35x-28y-21x+6y}{42}=\frac{14x-22y}{42}=\frac{7x-11y}{21}.$$

3. Simplify $\frac{1}{x^2+x}-\frac{1}{x^2-x}$.

We have, $x^2+x=x(x+1)$, and $x^2-x=x(x-1)$.

Then, the L. C. D. is $x(x+1)(x-1)$, or $x(x^2-1)$.

Multiplying the terms of the first fraction by $x-1$, and the terms of the second by $x+1$, we have

$$\frac{1}{x^2+x}-\frac{1}{x^2-x}=\frac{x-1}{x(x^2-1)}-\frac{x+1}{x(x^2-1)}$$

$$=\frac{x-1-(x+1)}{x(x^2-1)}=\frac{x-1-x-1}{x(x^2-1)}=\frac{-2}{x(x^2-1)}.$$

By changing the sign of the numerator, at the same time changing the sign before the fraction (§ 116), we may write the answer $-\frac{2}{x(x^2-1)}$.

Or, by changing the sign of the numerator, and of the factor x^2-1 of the denominator (§ 117), we may write it $\frac{2}{x(1-x^2)}$.

4. Simplify $\frac{1}{a^2-3a+2}-\frac{2}{a^2-4a+3}+\frac{1}{a^2-5a+6}$.

We have, $a^2-3a+2=(a-1)(a-2)$, $a^2-4a+3=(a-1)(a-3)$, and $a^2-5a+6=(a-2)(a-3)$.

Then, the L. C. D. is $(a-1)(a-2)(a-3)$.

Whence, $\frac{1}{a^2-3a+2}-\frac{2}{a^2-4a+3}+\frac{1}{a^2-5a+6}$

$$=\frac{a-3}{(a-1)(a-2)(a-3)}-\frac{2(a-2)}{(a-1)(a-2)(a-3)}+\frac{a-1}{(a-1)(a-2)(a-3)}$$

$$=\frac{a-3-2(a-2)+a-1}{(a-1)(a-2)(a-3)}=\frac{a-3-2a+4+a-1}{(a-1)(a-2)(a-3)}$$

$$=\frac{0}{(a-1)(a-2)(a-3)}=0.$$

EXERCISE 51

Simplify the following:

1. $\frac{2x+9}{8}+\frac{3x-5}{12}$.

2. $\frac{5}{4a^3c^2}+\frac{8}{7a^5c}$.

3. $\frac{2a-3b}{10}-\frac{3a-8b}{15}$.

4. $\frac{5R+2t}{6R^2t^2}-\frac{3R+8t}{9Rt^3}$.

5. $\frac{2c-7d^3}{13d^3}-\frac{5c+2d}{26c}$.

6. $\frac{4a+3b}{2ab}-\frac{c+2b}{3bc}+\frac{5a-c}{4ac}$.

7. $\frac{2(6n+5)}{11}-\frac{3(n+6)}{22}+\frac{4(5n-4)}{44}$.

8. $\frac{2a+3t}{14}-\frac{3a+2t}{21}+\frac{5a-7t}{28}$.

9. $\frac{3a^2-4}{10a^2}-\frac{4a^3+2}{5a^3}-\frac{6a^4-2}{25a^4}$.

10. $\frac{5x-4}{8x}-\frac{3y+2}{12y}-\frac{2z+5}{6z}$.

11. $\frac{5x-7}{5}-\frac{9x-8}{15}-\frac{12x-11}{20}+\frac{2x+9}{10}$.

12. $\frac{7t-4}{4}-\frac{3t-8}{5}-\frac{7t+7}{8}+\frac{6t-5}{10}$.

13. $\frac{2}{9}(3a+4b)-\frac{5}{18}(2a-5b)+\frac{7}{12}(a+2b)$.

14. $\frac{3}{3x-1}+\frac{2}{2x+1}$.

15. $\frac{1}{m+3}-\frac{2}{m-5}$.

16. $\frac{5}{m+5}+\frac{m}{m-3}$.

17. $\frac{x-3}{x-4}-\frac{x-4}{x-5}$.

18. $\frac{3x}{3x+y}-\frac{y}{3x-y}$.

19. $\frac{4}{4a-12}+\frac{5}{10a+15}$.

20. $\frac{5c}{5c-3}+\frac{c^2+8c-9}{c^2+4c-2}$.

21. $\frac{x}{2x-3}-\frac{x}{2x+3}-\frac{2x-6}{4x^2-9}$.

22. $\frac{3x+2}{3x-2}-\frac{9x^2+4}{9x^2-4}$.

23. $\frac{6a-5}{a^2-2a-15}+\frac{2a-1}{2a^2-5a-25}$.

24. $\frac{6a^2-4a}{a^3+27}-\frac{a-5}{6a^2+17a-3}$.

25. $\frac{5xy+7y^2}{3x-2y}-2y$.

Note: We may regard an integer as a fraction whose denominator is 1.

26. $\frac{2c-1}{3}+4-c-\frac{2c+1}{c+1}$.

27. $\frac{2m-1}{m-1}+\frac{4}{m+1}-\frac{3m}{m^2-1}-2$.

28. $\frac{3z+1}{5z-7}-\frac{2z^2-10z+12}{10z^2-49z+49}$.

29. $\frac{3t}{t+1}-\frac{15}{3t^2+t-2}+5$.

30. $\frac{c+d}{v}+\frac{c^2-d^2}{c^2+cd}-\frac{c}{v}+\frac{d}{c}$.

31. $\frac{3}{c-d}+\frac{2}{c+d}+5+\frac{1}{c+2d}$.

In certain cases, the principles of §§ 116 and 117 enable us to change the form of a fraction to one which is more convenient for the purposes of addition or subtraction.

32. $\frac{3}{a-b}+\frac{2b+a}{b^2-a^2}$.

Changing the signs of the terms in the second denominator, at the same time changing the sign before the fraction (§ 116) (see Exercise 46), we have $\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}$.

The L. C. D. is now a^2-b^2.

Then, $\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}=\frac{3(a+b)-(2b+a)}{a^2-b^2}$

$=\frac{3a+3b-2b-a}{a^2-b^2}=\frac{2a+b}{a^2-b^2}$.

33. $\frac{1}{(x-y)(x-z)}-\frac{1}{(y-x)(y-z)}-\frac{1}{(z-x)(z-y)}$.

By § 117, we change the sign of the factor $y-x$ in the second denominator, at the same time changing the sign before the fraction; and we change the signs of both factors of the third denominator.

The expression then becomes

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(x-y)(y-z)}-\frac{1}{(x-z)(y-z)}.$$

The L. C. D. is now $(x-y)(x-z)(y-z)$; then the result

$$=\frac{(y-z)+(x-z)-(x-y)}{(x-y)(x-z)(y-z)}=\frac{y-z+x-z-x+y}{(x-y)(x-z)(y-z)}.$$

$$=\frac{2y-2z}{(x-y)(x-z)(y-z)}=\frac{2(y-z)}{(x-y)(x-z)(y-z)}=\frac{2}{(x-y)(x-z)}.$$

34. $\dfrac{2}{3x-12}-\dfrac{3}{4-x}$.

35. $\dfrac{2a}{a^2-9}+\dfrac{5}{3-a}$.

36. $\dfrac{2e+3a}{2e-3a}+\dfrac{3e+4a}{4a-3e}$.

37. $\dfrac{x-1}{x^2-8x+15}+\dfrac{2-x}{x-5}$.

38. $\dfrac{v-b}{v-2a}-\dfrac{v+b}{v+2a}+\dfrac{b^2-4a^2}{4a^2-v^2}$.

39. $\dfrac{3x(a-b)}{x^2-b^2}-\dfrac{a-2b}{x+b}+\dfrac{a-b}{b-x}$.

40. $\dfrac{a}{k-a}-\dfrac{b}{k-b}-\dfrac{b^2-a^2}{b^2-bk}$.

41. $\dfrac{2(a+t)}{t}+\dfrac{a+t}{a-t}+\dfrac{t-a}{t+a}$.

42. $\dfrac{2u+7}{4-6u}+\dfrac{3u-5}{9u+6}-\dfrac{17u+2}{4-9u^2}$.

43. $\dfrac{m-2}{m-3}-\dfrac{3-m}{4-m}+\dfrac{m-5}{6-m}$.

MULTIPLICATION OF FRACTIONS

127. Required the product of $\frac{a}{b}$ and $\frac{c}{d}$.

Let $$\frac{a}{b}\cdot\frac{c}{d}=x. \tag{1}$$

(Multiplication may be indicated by either $\times$ or $\cdot$.)

Multiplying both members by $b\cdot d$ (Ax. 7, § 4),

$$\frac{a}{b}\cdot\frac{c}{d}\cdot b\cdot d=x\cdot b\cdot d, \text{ or } \left(\frac{a}{b}\cdot b\right)\left(\frac{c}{d}\cdot d\right)=x\cdot b\cdot d;$$

for the factors of a product may be written in any order.

Now since the product of the quotient and the divisor gives the dividend (§ 60), we have

$$\frac{a}{b}\cdot b=a, \text{ and } \frac{c}{d}\cdot d=c.$$

Whence, $$(a)(c)=x\cdot b\cdot d.$$

Dividing both members by $b\cdot d$ (Ax. 8, § 4),

$$\frac{a\cdot c}{b\cdot d}=x. \tag{2}$$

From (1) and (2), $$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

To multiply fractions, multiply the numerators together for the numerator of the product, and the denominators for its denominator.

128. Since c may be regarded as a fraction having the denominator 1, we have, by § 127,

$$\frac{a}{b} \cdot c = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b}.$$

Dividing both numerator and denominator by c (§ 115),

$$\frac{a}{b} \cdot c = \frac{a}{b \div c}.$$

Then, to multiply a fraction by a rational and integral expression, if possible, divide the denominator of the fraction by the expression; otherwise, multiply the numerator by the expression.

129. Common factors in the numerators and denominators should be cancelled before performing the multiplication.

Mixed expressions should be expressed in a fractional form (§ 122) before applying the rules.

1. Multiply $\frac{10\,a^3y}{9\,bx^2}$ by $\frac{3\,b^4x^3}{4\,a^3y^2}$.

$$\frac{10\,a^3y}{9\,bx^2} \cdot \frac{3\,b^4x^3}{4\,a^3y^2} = \frac{2 \cdot 5 \cdot 3 \cdot a^3b^4x^3y}{3^2 \cdot 2^2 \cdot a^3bx^2y^2} = \frac{5\,b^3x}{6\,y}.$$

The factors cancelled are 2, 3, a^3, b, x^2, and y.

2. Multiply together $\frac{x^2+2x}{x^2+x-6}$, $2-\frac{x-4}{x-3}$, and $\frac{x^2-9}{x^2-4}$.

$$\frac{x^2+2x}{x^2+x-6} \cdot \left(2-\frac{x-4}{x-3}\right) \cdot \frac{x^2-9}{x^2-4}$$

$$= \frac{x^2+2x}{x^2+x-6} \cdot \frac{2x-6-x+4}{x-3} \cdot \frac{x^2-9}{x^2-4}$$

$$= \frac{x(x+2)}{(x+3)(x-2)} \cdot \frac{x-2}{x-3} \cdot \frac{(x+3)(x-3)}{(x+2)(x-2)} = \frac{x}{x-2}.$$

The factors cancelled are $x+2$, $x-2$, $x+3$, and $x-3$.

3. Multiply $\frac{a^2+b^2}{a^2-b^2}$ by $a-b$.

Dividing the denominator by $a-b$, $\frac{a^2+b^2}{a^2-b^2} \cdot (a-b) = \frac{a^2+b^2}{a+b}$.

4. Multiply $\frac{m}{m-n}$ by $m+n$.

Multiplying the numerator by $m+n$, $\frac{m}{m-n} \cdot (m+n) = \frac{m^2+mn}{m-n}$.

EXERCISE 52

Simplify the following:

1. $\frac{8\,am^2}{27\,b^4n^5} \times 9\,bn^5.$

2. $\frac{21\,a^3b^2}{8\,cd^6} \times \frac{4\,c^5d^6}{35\,a^3b^5}.$

3. $\frac{5\,a^2}{3\,b^2} \times \frac{9\,b^3}{10\,c^3} \times \frac{7\,c^4}{6\,a^4}.$

4. $\frac{3\,x^5}{10\,y^3} \cdot \frac{15\,y^7}{7\,z} \cdot \frac{28\,z^3}{9\,x^2}.$

5. $\frac{14\,b^3c}{15\,a^6} \cdot \frac{5\,c^3a}{12\,b^5} \cdot \frac{6\,a^2b}{7\,c^4}.$

6. $\frac{28\,m^7}{25\,n^5x^3} \cdot \frac{15\,n^6}{14\,m^5x^2} \cdot \frac{5\,x^3}{21\,m^3n^4}.$

7. $\frac{a^2-ab}{a^2+4\,a+4} \cdot (a+2).$

8. $\frac{5\,c+x}{x^2+4\,x-12} \cdot (x-2).$

9. $\frac{35\,a^2b}{4\,n^2-36\,d^2} \cdot \frac{n^2-6\,nd+9\,d^2}{20\,ab^2}.$

10. $\frac{a^2-2\,a-35}{2\,a^3-3\,a^2} \cdot \frac{4\,a^3-9\,a}{a-7}.$

11. $\frac{16\,z^2-9\,y^2}{8\,z^2+22\,zy-21\,y^2} \cdot \frac{2\,z^2+11\,zy+14\,y^2}{4\,z^2+11\,zy+6\,y^2}.$

12. $\frac{4\,t^2+4\,t+1}{3\,t+5} \cdot \frac{3\,tc+5\,c+6\,td+10\,d}{4\,t^2+10\,t+4}.$

13. $\frac{a^3-8\,b^3}{a^2-4\,b^2} \cdot \frac{a^2+4\,ab+4\,b^2}{2\,a^3+4\,a^2b+8\,ab^2}.$

14. $\left(4-\frac{a^2+2\,ab+b^2}{ab}\right)\left(\frac{4\,ab}{a^2-2\,ab+b^2}+1\right).$

15. $\frac{3\,x-4\,y}{3\,x+4\,y} \cdot \frac{9\,x^2+24\,xy+16\,y^2}{3\,x^2-4\,xy}+2.$

Note: In problems similar to example 15, indicated multiplication or division must be performed before addition or subtraction is made. $2\,x$ is to be added to the product of the fractions, not to the second fraction.

For example, in $13+4\times3+6\div2-4$, 4×3 and $6\div2$ must be performed before uniting the terms of the expression.

16. $\frac{25\,m^2-40\,m+16}{4\,m^2-9} \cdot \frac{8\,m^3-12\,m^2}{25\,m^2-16}-2\,m+\frac{8}{2\,m+3}.$

DIVISION OF FRACTIONS

130. Required the quotient of $\frac{a}{b}$ divided by $\frac{c}{d}$.

Let $$\frac{a}{b} \div \frac{c}{d} = x. \tag{1}$$

Then since the dividend is the product of the divisor and quotient (§ 60), we have $$\frac{a}{b} = \frac{c}{d} \times x.$$

Multiplying both members by $\frac{d}{c}$ (Ax. 7, § 4),

$$\frac{a}{b} \times \frac{d}{c} = \frac{c}{d} \times x \times \frac{d}{c} = x. \tag{2}$$

From (1) and (2), $$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$ (Ax. 4, § 4)

Then, to divide one fraction by another, multiply the dividend by the divisor inverted.

If the divisor is an integer, c may be regarded as a fraction having the denominator 1.

Mixed expressions should be expressed in a fractional form (§ 122) before applying the rules.

1. Divide $\frac{6\,a^2b}{5\,x^3y^4}$ by $\frac{9\,a^2b^3}{10\,x^2y^7}$.

We have, $$\frac{6\,a^2b}{5\,x^3y^4} \div \frac{9\,a^2b^3}{10\,x^2y^7} = \frac{6\,a^2b}{5\,x^3y^4} \cdot \frac{10\,x^2y^7}{9\,a^2b^3} = \frac{4\,y^3}{3\,b^2x}.$$

2. Divide $2 - \frac{2\,x-3}{x+1}$ by $3 - \frac{3\,x^2-13}{x^2-1}$.

$$\left(2 - \frac{2\,x-3}{x+1}\right) \div \left(3 - \frac{3\,x^2-13}{x^2-1}\right)$$

$$= \frac{2\,x+2-2\,x+3}{x+1} \div \frac{3\,x^2-3-3\,x^2+13}{x^2-1}$$

$$= \frac{5}{x+1} \cdot \frac{x^2-1}{10} = \frac{5(x+1)(x-1)}{2 \cdot 5 \cdot (x+1)} = \frac{x-1}{2}.$$

3. Divide $\frac{m^3-n^3}{m^2+n^2}$ by $m-n$.

Dividing the numerator by $m-n$, $\frac{m^3-n^3}{m^2+n^2} \div (m-n) = \frac{m^2+mn+n^2}{m^2+n^2}$.

4. Divide $\frac{a^2+b^2}{a-b}$ by $a+b$.

Multiplying the denominator by $a+b$, $\frac{a^2+b^2}{a-b} \div (a+b) = \frac{a^2+b^2}{a^2-b^2}$.

If the numerator and denominator of the divisor are exactly contained in the numerator and denominator, respectively, of the dividend, it follows from § 127 that the numerator of the quotient may be obtained by dividing the numerator of the dividend by the numerator of the divisor; and the denominator of the quotient by dividing the denominator of the dividend by the denominator of the divisor.

5. Divide $\frac{9x^2-4y^2}{x^2-y^2}$ by $\frac{3x+2y}{x-y}$.

We have, $\frac{9x^2-4y^2}{x^2-y^2} \div \frac{3x+2y}{x-y} = \frac{3x-2y}{x+y}$.

EXERCISE 53

Simplify the following:

1. $\frac{45x^6m^2}{4y^5n} \div 9x^3m^4$.

2. $\frac{12a^6b^2}{55c^3d^5} \div \frac{9a^4b^8}{22c^7d^5}$.

3. $\frac{9x^2-16}{3x+7} \div (3x-4)$.

4. $\frac{4m-k}{2m+3k} \div (4m+k)$.

5. $\frac{t^2-t-12}{8t} \div \frac{t^2-8t+16}{6t}$.

6. $\frac{2v^2-5v-12}{3v+2} \div \frac{4v^2-10v-24}{9v^2-4}$.

7. Divide $\frac{2ac-2bc+3ad-3bd}{4c^2+4cd+d^2}$ by $\frac{a^2+5ab-6b^2}{4c^2-4cd-3d^2}$.

8. $\frac{a^3+27}{a^4+4} \div \frac{a^2b-3ab+9b}{a^2+2a+2}$.

9. $\frac{5t^2+8tu+3u^2}{t^3+u^3} \div \frac{3t^2+7tu+4u^2}{t+u}$.

10. $2-\frac{3x-3}{4x+7} \div 3-\frac{4x-26}{3x-3}$.

11. Divide $2-\frac{3x-3}{4x+7}$ by $3-\frac{4x-26}{3x-3}$.

12. $\left(\frac{4t-3}{t+2}-5t+2\right) \div \left(\frac{4t^2-2t-9}{t+2}+t+4\right)$.

COMPLEX FRACTIONS

131. A Complex Fraction is a fraction having one or more fractions in either or both of its terms.

It is simply a case in division of fractions; its numerator being the dividend, and its denominator the divisor.

1. Simplify $\dfrac{a}{b-\dfrac{c}{d}}$.

$$\frac{a}{b-\frac{c}{d}}=\frac{a}{\frac{bd-c}{d}}=a\times\frac{d}{bd-c}\ (\S\,130)=\frac{ad}{bd-c}.$$

It is often advantageous to simplify a complex fraction by multiplying its numerator and denominator by the L. C. M. of their denominators (§ 115).

2. Simplify $\dfrac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}$.

The L. C. M. of $a+b$ and $a-b$ is $(a+b)(a-b)$.
Multiplying both terms by $(a+b)(a-b)$, we have

$$\frac{\frac{a}{a-b}-\frac{a}{a+b}}{\frac{b}{a-b}+\frac{a}{a+b}}=\frac{a(a+b)-a(a-b)}{b(a+b)+a(a-b)}=\frac{a^2+ab-a^2+ab}{ab+b^2+a^2-ab}=\frac{2\,ab}{a^2+b^2}.$$

EXERCISE 54

Simplify the following:

1. $\dfrac{\frac{1}{r}+\frac{1}{s}}{\frac{1}{r}-\frac{1}{s}}$.

2. $\dfrac{\frac{r^2+u^2}{ru}}{\frac{1}{r}+\frac{1}{u}}$.

3. $\dfrac{1-\frac{3\,a}{7\,b}}{b-\frac{9\,a^2}{49\,b}}$.

4. $\dfrac{c+\frac{8\,d^3}{c^2}}{1+\frac{2\,d}{c}}$.

5. $\dfrac{\frac{x-y}{x}-\frac{x+y}{y}}{\frac{x+y}{x}+\frac{x-y}{y}}$.

6. $\dfrac{\frac{2\,a}{3\,c}+2+\frac{3\,c}{2\,a}}{\frac{1}{3\,c}+\frac{1}{2\,a}}$.

7. $\dfrac{\dfrac{a^2-2a-35}{2a^3-3a^2}}{\dfrac{a-7}{4a^3-9a}}$.

8. $\dfrac{\dfrac{4m^2+8m+3}{2m^2-5m+3}}{\dfrac{4m^2-1}{6m^2-9m}}$.

9. $\dfrac{\dfrac{1}{x+2}+\dfrac{2}{x+3}}{\dfrac{x^2-3x-10}{x+2}}$.

10. $\dfrac{\dfrac{x-2}{x+3}-\dfrac{x+2}{x-3}}{\dfrac{x}{x+3}-\dfrac{x}{x-3}}$.

11. $\dfrac{\dfrac{a^2-b^2-c^2}{2bc}-1}{\dfrac{a^2+b^2-c^2}{2ab}+1}$.

12. $\dfrac{\dfrac{e^4+e^2f^2+f^4}{e^3+f^3}}{\dfrac{e^2+ef+f^2}{e^2-f^2}}$.

13. $\dfrac{\dfrac{4v^2-23v+6}{v^2-7v+12}-2}{3-\dfrac{5v^2-72}{v^2+3v-18}}$.

14. $\dfrac{\left(x-\dfrac{3}{x}\right)^2+16}{\left(x+\dfrac{3}{x}\right)^2+4}$.

15. $\dfrac{\dfrac{x^2-y^2-2yz-z^2}{9t^2-25u^2}}{\dfrac{(x+y+z)(x-y-z)}{6t+10u}}$.

16. $\dfrac{\dfrac{5h^2+14hk-3k^2}{8h^2-6hk-35k^2}}{\dfrac{5h^2-11hk+2k^2}{4h^2-hk-14k^2}}$.

MISCELLANEOUS AND REVIEW EXAMPLES

EXERCISE 55

Simplify the following:

1. $\dfrac{3a-2b}{a^2-4b^2}\cdot(a-2b)$.

2. $\dfrac{8c^3+27d^3}{8c^3-27d^3}\cdot(2c-3d)$.

3. $\left(6a^2+\dfrac{b}{5}\right)\div\left(c^2-\dfrac{x-a}{2}\right)$.

4. $x+\dfrac{x+1}{a}\cdot\dfrac{x-1}{a+b}$.

5. $\left(x+\dfrac{x+1}{a}\right)\left(\dfrac{x-1}{a+b}\right)$.

6. $\dfrac{x^4-b^4}{x^2-2bx+b^2}\div\dfrac{x^2+bx}{x-b}$.

7. From $3a+\dfrac{a}{b}$ take $a-\dfrac{a-c}{d}$.

8. Find the sum of $c-\frac{6\,b}{8\,a}$ and $1-\frac{c}{a}$.

9. Find the sum of $3\,a+\frac{2\,ax}{x-a}$ and $2\,a-\frac{-x}{a+b}$.

Simplify:

10. $a-b-\frac{a^2+b^2}{a-b}$.

11. $a+b-\frac{(a-b)^2}{a+b}$.

12. $a+b+\frac{(a-b)^2}{a+b}$.

13. $\frac{x^4-3\,x^3-8\,x+24}{x^4+3\,x^3-8\,x-24}$.

14. $\frac{(a+2\,c)^2-(b-3)^2}{(2\,c+3)^2-(a-b)^2}$.

15. $\frac{\frac{1}{3\,x}+\frac{1}{2\,y}}{9\,x^2-4\,y^2}+\frac{\frac{1}{2\,x}-\frac{1}{3\,y}}{4\,x^2-9\,y^2}$.

16. $\frac{x(x^2+x+1)-2(x^2+x+1)}{x^4+x^2+1}$.

17. $\left(\frac{a+2}{a}+\frac{2}{a-3}\right)\left(\frac{a}{a-2}-\frac{3}{a+3}\right)$.

18. $\frac{1+e^2+\frac{e^4}{1-e^2}}{1-e^2}$.

19. $\frac{1+\frac{c^2}{x^2-a^2}}{x^2-a^2+c^2}$.

20. $\frac{\frac{a}{x^2}}{1+\frac{a^2}{x^2}}+\frac{\frac{a}{(x-a)^2}}{\frac{x+a}{x-a}}$.

21. $\frac{1}{x-1}-\frac{1}{x-2}+\frac{1}{x-3}-\frac{1}{x-4}$.

(Combine the first two fractions, then the last two, and add the results.)

22. $\frac{3}{2\,n+1}+\frac{3}{2\,n-1}-\frac{5\,n^2}{8\,n^3+1}-\frac{5\,n^2}{8\,n^3-1}$.

23. $\frac{a}{a+3}+\frac{1}{a-3}-\frac{3}{a^2-9}-\frac{a^2+2\,a}{a^2+9}$.

(First add the first two fractions, to the result add the third fraction and to this result add the last fraction.)

24. $\dfrac{3a}{a+b}+\dfrac{3a}{a-b}+\dfrac{6a^2}{a^2+b^2}+\dfrac{12a^4}{a^4+b^4}$.

25. $\dfrac{1}{x^2+2x-3}-\dfrac{1}{x^2+x-6}+\dfrac{1}{x^2-3x+2}$.

26. $\dfrac{x-2}{2x^2-13x-45}-\dfrac{x+3}{2x^2+29x+60}+\dfrac{4x-1}{x^2+3x-108}$.

27. Translate into English:

$$\frac{4a-b}{2a+b}-5c\times 2a-\frac{3a+b}{c}.$$

28. Translate into English:

$$\left(\frac{4a-b}{2a+b}-5c\right)\left(2a-\frac{3a+b}{c}\right).$$

29. Translate into English:

$$\frac{3m-1}{2a-1}-4\div 2-\frac{4m-1}{3a-1}.$$

30. Translate into English:

$$\left(\frac{3m-1}{2a-1}-4\right)\div\left(2-\frac{4m-1}{3a-1}\right).$$

31. State algebraically: The sum of $3a$ and b divided by the difference between $3a$ and b: Multiply this quotient by the fraction whose numerator is the difference between $9a^2$ and b^2 and whose denominator is the sum of $9a^2$ and b^2. Reduce your statement.

X. FRACTIONAL EQUATIONS. RATIO AND PROPORTION

SOLUTION OF FRACTIONAL EQUATIONS

132. If a fraction whose numerator is a polynomial is preceded by a $-$ sign, it is convenient, on clearing of fractions, to write the numerator in parenthesis, as shown in Ex. 1.

If this is not done, care must be taken to *change the sign of each term of the numerator* when the denominator is removed. This is readily understood if one remembers that the line between the numerator and denominator acts as a *vinculum*.

1. Solve the equation $\frac{3t-1}{4}-\frac{4t-5}{5}=4+\frac{7t+5}{10}$.

The L. C. M. of 4, 5, and 10 is 20.
Multiplying each term by 20, we have

$$15t-5-(16t-20)=80+14t+10.$$

Whence, $15t-5-16t+20=80+14t+10.$
Transposing, $15t-16t-14t=80+10+5-20.$
Uniting terms, $-15t=75.$
Dividing by -15, $t=-5.$
Verify the result.

2. Solve the equation $\frac{2}{x-2}-\frac{5}{x+2}-\frac{2}{x^2-4}=0.$

The L. C. M. of $x-2$, $x+2$, and x^2-4 is x^2-4.
Multiplying each term by x^2-4, we have

$$2(x+2)-5(x-2)-2=0.$$

Or, $2x+4-5x+10-2=0.$
Transposing, and uniting terms, $-3x=-12$, and $x=4$.
Verify the result.

If the denominators are partly monomial and partly polynomial, it is often advantageous to clear of fractions at first partially; multiplying each term of the equation by the L. C. M. of the *monomial* denominators.

3. Solve the equation $\frac{6s+1}{15}-\frac{2s-4}{7s-16}=\frac{2s-1}{5}$.

Multiplying each term by 15, the L. C. M. of 15 and 5,

$$6s+1-\frac{30s-60}{7s-16}=6s-3.$$

Transposing, and uniting terms, $4=\frac{30s-60}{7s-16}.$
Clearing of fractions, $28s-64=30s-60.$
Then, $-2s=4$, and $s=-2.$
Verify the result.

EXERCISE 56

Solve the following equations, verifying each result:

1. $\frac{1}{3}+\frac{2}{5x}=\frac{13}{15}-\frac{2}{3x}.$

2. $\frac{1}{x}+\frac{2}{3x}-\frac{3}{5x}-\frac{2}{15x}=\frac{7}{30}$.

3. $\frac{4}{x}-\frac{1}{4x}-\frac{6}{7x}-\frac{2}{3x}=\frac{187}{168}$.

4. $3x-\frac{5x+15}{6}=\frac{4x}{3}$.

5. $\frac{5}{3}t-\frac{4t-11}{5}=3t+\frac{1}{15}$.

6. $\frac{4v}{5}+\frac{2v-7}{4}+\frac{1}{12}=\frac{7v}{15}$.

7. $R-\frac{4R+7}{3}+\frac{5R+9}{8}=-\frac{3}{2}$.

8. $\frac{3m-1}{4m}-\frac{5m+1}{8m}+\frac{9}{40}-\frac{8m-5}{5m}=0$.

9. $\frac{5}{u}-\frac{1}{21}-\frac{5(11-3u)}{6u}=\frac{7-9u}{2u}-\frac{5}{7u}$.

10. $\frac{5(x-1)}{6}-\frac{2(x+2)}{3}=4-\frac{5x-15}{4}$.

11. $\frac{6g+4}{8}-\frac{3g-4}{2}+1=\frac{5g+8}{9}$.

12. $\frac{2v+1}{7}-\frac{6v-4}{9v+1}=\frac{4v-5}{14}$.

13. $\frac{8x+15}{3}-\frac{11x+15}{5}-\frac{13x+29}{10}=\frac{14x+66}{15}$.

14. $\frac{2t+9}{4t+1}=\frac{t-11}{2t-15}$.

15. $\frac{3x-1}{18x-19}=\frac{x+1}{6x-7}$.

16. $\frac{24}{x-2}-\frac{24}{x}=1$. (See § 103.)

17. $\frac{t-2}{t+5}-\frac{t+4}{t-3}=-\frac{7}{3}$.

18. $\frac{5}{5-R}+\frac{8}{8-R}=3$.

19. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-3}{x-4}-\frac{x-4}{x-5}$

(Reduce each fraction to a mixed number, then see example 21, Exercise 55.)

20. $\dfrac{2x}{x+1}+\dfrac{2x+1}{x}=\dfrac{3x+2}{x^2+x}$.

21. What number added to twice its reciprocal gives 3?

22. The sum of $\frac{1}{3}$ of a certain number and $\frac{1}{3}$ of its square is $\frac{1}{4}$. Find the number.

23. Make two problems similar to 21 and 22.

133. Solution of Special Forms of Fractional Equations.

1. Solve the equation $\dfrac{2x-1}{2x-3}+\dfrac{x^2-x}{x^2+4}=2$.

We divide each numerator by its corresponding denominator; then

$$1+\frac{2}{2x-3}+1-\frac{x+4}{x^2+4}=2, \text{ or } \frac{2}{2x-3}-\frac{x+4}{x^2+4}=0.$$

Clearing of fractions, $2x^2+8-(2x^2+5x-12)=0$.

Then, $2x^2+8-2x^2-5x+12=0$; whence, $x=4$.

We reject a solution which does not satisfy the given equation.

2. Solve the equation $\dfrac{1}{x-3}+\dfrac{1}{x-2}=\dfrac{3x-7}{x^2-5x+6}$.

Multiplying both members by $(x-3)(x-2)$, or x^2-5x+6,

$$x-2+x-3=3x-7.$$

Transposing, and uniting terms, $-x=-2$, or $x=2$.

If we substitute 2 for x, the fraction $\dfrac{1}{x-2}$ becomes $\dfrac{1}{0}$.

Since division by 0 is impossible, the solution $x=2$ does not satisfy the given equation, and we reject it; the equation has no solution.

3. Solve the equation $\dfrac{3}{x+10}+\dfrac{4}{x+6}=\dfrac{2}{x+8}+\dfrac{5}{x+9}$.

Adding the fractions in each member, we have

$$\frac{7x+58}{(x+10)(x+6)}=\frac{7x+58}{(x+8)(x+9)}.$$

Clearing of fractions, and transposing all terms to the first member,

$$(7x+58)(x+8)(x+9)-(7x+58)(x+10)(x+6)=0. \qquad (1)$$

Factoring, $(7x+58)[(x+8)(x+9)-(x+10)(x+6)]=0$.

Expanding, $(7x+58)(x^2+17x+72-x^2-16x-60)=0$.

Or, $(7x+58)(x+12)=0$.

This equation may be solved by the method of § 103.

Placing $7x+58=0$, we have $x=-\dfrac{58}{7}$.

Placing $x+12=0$, we have $x=-12$.

134. If we should solve equation (1), in Ex. 3 of § 133, by dividing both members by $7x+58$, we should have

$$(x+8)(x+9)-(x+10)(x+6)=0.$$

Then, $x^2+17x+72-x^2-16x-60=0$, or $x=-12$.

In this way, the solution $x=-\frac{58}{7}$ *is lost.*

It follows from this that it is never allowable to divide both members of an equation by any expression which involves the unknown numbers, unless the expression be placed equal to 0 and the root preserved, for in this way solutions are lost.

EXERCISE 57

Solve the following equations:

1. $\dfrac{4x+11}{x^2+x-20}=\dfrac{1}{x+5}-\dfrac{1}{x-4}.$

2. $\dfrac{x+3}{x+2}+\dfrac{x+4}{x+3}+\dfrac{x+2}{x+4}=3.$

3. $\dfrac{3}{x+9}+\dfrac{2}{x+4}=\dfrac{1}{x+3}+\dfrac{4}{x+18}.$

4. $\dfrac{2x+3}{2x-3}-\dfrac{2x-3}{2x+3}-\dfrac{36}{4x^2-9}=0.$

5. $\dfrac{2x+5}{x+7}-\dfrac{3x^2+24x+19}{x^2+8x+7}=-1.$

6. $\dfrac{x^2-2x+5}{x^2-2x-3}+\dfrac{x^2+3x-7}{x^2+3x+1}=2.$

SOLUTION OF LITERAL LINEAR EQUATIONS

135. A **Literal Equation** is one in which some or all of the known numbers are represented by letters; as,

$$2x+a=b^2+10.$$

Ex. Solve the equation $\dfrac{x}{x-a}-\dfrac{x+2b}{x+a}=\dfrac{a^2+b^2}{x^2-a^2}.$

Multiplying each term by x^2-a^2,

$$x(x+a)-(x+2b)(x-a)=a^2+b^2,$$

or, $$x^2+ax-(x^2+2bx-ax-2ab)=a^2+b^2,$$

or, $$x^2+ax-x^2-2bx+ax+2ab=a^2+b^2,$$

or, $$2ax-2bx=a^2-2ab+b^2.$$

Factoring both members, $$2x(a-b)=(a-b)^2.$$

Dividing by $2(a-b)$, $$x=\frac{(a-b)^2}{2(a-b)}=\frac{a-b}{2}.$$

In solving fractional literal equations, we must reject any solution which does not satisfy the given equation. Compare Ex. 2, § 133.

EXERCISE 58

1. Find the coefficient of x in

$$(b+x)^2+(2c-3ax)^2.$$

2. Find the coefficient of t^2 in

$$a(t-b)(t-b)-b(t-a)(t-a)-3at(2a+t).$$

3. $\frac{5x-a}{2x+a}+\frac{2x+5a}{4x}=3$. Solve for x.

4. $\frac{v}{ab}+\frac{v}{bc}+\frac{v}{ca}=a+b+c$. Solve for v.

5. $\frac{c^2t+b}{ct}-\frac{b^2t+c}{bt}=\frac{c^3b-cb^3+b^2-c^2}{cbt}$. Solve for t.

6. $\frac{2m+3b}{2m-3b}-\frac{4a+5b}{4a-5b}=0$. Solve for m.

7. $\frac{u(a+4b)-b^2}{a^2-b^2}+\frac{u-b}{a+b}=\frac{u+a}{a-b}$. Solve for u.

8. $\frac{a-b}{t-c}+\frac{b-c}{t-a}+\frac{c-a}{t}=0$. Solve for t.

9. $\frac{5}{2s+5d}-\frac{2}{3s-4d}=\frac{3d}{6s^2+7ds-20d^2}$. Solve for s.

10. $\frac{a}{w-a}-\frac{b}{w-b}=\frac{b^2-a^2}{b^2-bw}$. Solve for w.

11. $\frac{a(x-a)}{x-b}+\frac{b(x-b)}{x-a}=a+b$. Solve for x.

12. $\frac{4v+3n}{v+2n}+\frac{4v-3n}{3n-v}=-\frac{10n^2}{v^2-nv-6n^2}$. Solve for n.

13. $\frac{3x}{2}-\frac{5ax-2b}{4a}=\frac{a+3bx}{8b}-\frac{ax+2a^2-4b}{16ab}$.

14. $\frac{2R+3a}{2R-3a}=\frac{3a+4b}{3a-4b}$. Solve for R.

15. $\frac{ax-b}{bx}+\frac{bx+a}{ax}=2+\frac{a-b}{abx}$. Solve for x.

RATIO AND PROPORTION

RATIO

136. *The* **Ratio** *of one whole or fractional number*, **a**, *to another*, **b**, *is the quotient of* **a** *divided by* **b**. Thus, the ratio of a to b is $\frac{a}{b}$; it is also expressed $a:b$. We make no attempt to define *ratio*. When applied to whole or fractional numbers, ratio is only another name for quotient or fraction. When the fraction $\frac{a}{b}$ is called a ratio, its numerator a is called the *antecedent* or *first term*, and its denominator b is called the *consequent* or *second term*.

The ratios here spoken of are but fractions under another name, and *have all the properties of fractions*.

If a and b are positive numbers, and $a>b$, $\frac{a}{b}$ is called a *ratio of greater inequality*; if $a<b$, it is called a *ratio of less inequality*.

(The signs $>$ and $<$ are read "is greater than" and "is less than" respectively.)

PROPORTION

137. A Proportion is an equation whose members are equal ratios.

Thus, if $\frac{a}{b}$ and $\frac{c}{d}$ are equal ratios,

$$a:b=c:d, \text{ or } \frac{a}{b}=\frac{c}{d},$$

is a proportion. (See example 14, Exercise 58.)

138. In the above proportion, a is called the *first term*, b the *second*, c the *third*, and d the *fourth*.

The first and third terms of a proportion are called the *antecedents*, and the second and fourth terms the *consequents*.

The first and fourth terms are called the *extremes*, and the second and third terms the *means*.

139. If the means of a proportion are equal, either mean is called the **Mean Proportional** between the first and last terms, and the last term is called the **Third Proportional** to the first and second terms.

Thus, in the proportion $\frac{a}{b}=\frac{b}{c}$, b is the mean proportional between a and c, and c is the third proportional to a and b.

The **Fourth Proportional** to three numbers is the fourth term of a proportion whose first three terms are the three numbers taken in their order.

Thus, in the proportion $\frac{a}{b}=\frac{c}{d}$, d is the fourth proportional to a, b, and c.

140. A **Continued Proportion** is a series of equal ratios, in which each consequent is the same as the next antecedent; as,

$$a : b = b : c = c : d = d : e$$

or

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}.$$

The definitions and explanations in §§ 138 and 139 refer to proportions written in the form

$$a : b = c : d.$$

Because of greater facility in operation, however, we shall use the form

$$\frac{a}{b}=\frac{c}{d}.$$

IMPORTANT PROPERTIES OF PROPORTIONS

141. In any proportion, the product of the extremes is equal to the product of the means.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$.

Clearing of fractions, $ad=bc$.

142. (Converse of § 141.) If the product of two numbers be equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.

Let $ad=bc.$

Dividing by bd, $\frac{ad}{bd}=\frac{bc}{bd}$, or $\frac{a}{b}=\frac{c}{d}$.

143. In any proportion, the terms are in proportion by Alternation; that is, the means can be interchanged.

In § 142, had we divided by cd, the proportion would have been

$$\frac{a}{c}=\frac{b}{d}.$$

In like manner, the extremes can be interchanged.

144. In any proportion, the terms are in proportion by Inversion; that is, the second term is to the first as the fourth term is to the third.

It follows from § 144 that, in any proportion, the means can be written as the extremes, and the extremes as the means.

145. In any proportion, the terms are in proportion by Composition; that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$. Then $\frac{a+b}{a}=\frac{c+d}{c}$,

also, $\frac{a+b}{b}=\frac{c+d}{d}$.

146. In any proportion, the terms are in proportion by Division; that is, the difference between the first two terms is to the first term as the difference between the last two terms is to the third term.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$. Then $\frac{a-b}{a}=\frac{c-d}{c}$,

also, $\frac{a-b}{b}=\frac{c-d}{d}$.

147. In any proportion, the terms are in proportion by Composition and Division; that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.

Let the proportion be $\frac{a}{b}=\frac{c}{d}$. Then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

148. In any number of proportions, the products of the corresponding terms are in proportion.

Let the proportions be $\frac{a}{b}=\frac{c}{d}$, and $\frac{e}{f}=\frac{g}{h}$.

Then, $\frac{ae}{bf}=\frac{cg}{dh}$.

149. In a series of equal ratios, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let, $a:b=c:d=e:f$.

Then, $\frac{a}{b}=\frac{a+c+e}{b+d+f}$.

EXERCISE 59

The following problems lead both to integral and fractional equations. *Always verify results.* In verifying results obtained from written problems it is sufficient to ascertain if results satisfy the conditions stated in the problem.

1. m is a mean proportional between 2 and 8; find m.

2. x is a positive integer. If x be added to both terms of the ratio $\frac{3}{5}$, what is the effect on the ratio? If $\frac{5}{3}$ were the ratio would the effect be the same?

3. In any proportion if the first antecedent and its consequent be multiplied by m, is the proportion changed? Why?

4. d is a fourth proportional to 3, 4, and 12; find d.

5. A can do in 8 days a piece of work which B can perform in 10 days. In how many days can it be done by both working together?

Let x = the number of days required.

Then, $\frac{1}{x}$ = the part both can do in one day.

Also, $\frac{1}{8}$ = the part A can do in one day,

and $\frac{1}{10}$ = the part B can do in one day.

By the conditions, $\frac{1}{8}+\frac{1}{10}=\frac{1}{x}$.

Clearing of fractions, $5x+4x=40$, or $9x=40$.

Whence, $x=4\frac{4}{9}$, the number of days required.

6. A piece of work can be done by A in 7 hours, and by B in 6 hours; in how many hours can the work be done by both working together?

7. The numerator of a certain fraction is 4 greater than the denominator. If 5 be added to both numerator and denominator, the result is $\frac{3}{2}$. Find the fraction.

(Hint: Let d = the denominator.)

8. Given two numbers, 5 and 2. What number must you add to each so that the first sum may be $\frac{4}{5}$ of the second sum?

9. A's age is 4 years more than $\frac{3}{5}$ of B's, and the sum of their ages is 44. Find the age of each.

Write in the form of proportion:

10. $x^2+3x+2=a^2-a-12$.

11. $x^2-6x+9=a^2-9$.

12. $4c^2+12c+9=d^2-4d+4$.

13. $x(x+3)=y(y-7)$.

14. $x(y-3)=y(x-3)$.

15. Solve, using composition and division:

$$\frac{2t-1}{2t+1}=\frac{a-b}{a+b}.$$

16. Use composition and division, then solve:

$$\frac{2x+3}{2x-4}=\frac{x+2}{x-2}.$$

17. Write this proportion as a simple equation:

$$\frac{x-4}{y+2}=\frac{a+2}{a-4}.$$

18. The first digit of a number exceeds the second by 3; and if the number, increased by 4, be divided by the sum of its digits, the quotient is 8. Find the number.

(Let x = the second digit, the number itself is ten times the first digit, plus the second digit.)

19. A piece of work can be done by A and B working together in 10 days. After working together 7 days, A leaves,

and B finishes the work in 9 days. How long would A alone have taken to do the work?

20. Solve this proportion: $\frac{x-1}{x+1}=\frac{2}{3}$.

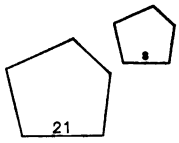
21. Simplify, using composition and division:

$$\frac{a-1}{a+1}=\frac{c-1}{c+1}.$$

22. $\frac{t+(a+b)}{t-(a+b)}=\frac{a}{b}$. Solve for t.

23. The volumes of two spheres are proportional to the cubes of their like dimensions. If a sphere 6 inches in diameter weighs 351 ounces, what is the weight of a sphere of the same material whose diameter is 10 inches?

24. In similar figures in geometry, like lines are proportional. The sums of the sides of two similar polygons are 119 and 68 respectively. If a side of the first polygon is 21, what is the corresponding side of the second?



25. A garrison of 700 men has provisions for 11 days. After 3 days a certain number of men leave, and the provisions last 10 days after this time. How many men leave?

26. The digits of a certain number are three consecutive numbers, of which the middle digit is the least, and the last digit is the greatest. If the number be divided by the sum of the digits the quotient is 36. What is the number?

27. If b is a mean proportional between a and c, show that $\frac{a}{c}=\frac{a^2}{b^2}$.

28. The angles of a triangle ABC are in proportion to 1, 2, 3. The sum of the angles of a triangle is 180°. How many degrees in each angle? See note, Ex. 31.

29. How many degrees in each angle of the triangle ABC, if angle B is twice angle A, and angle C is 20° more than B?

30. B can do a piece of work in $\frac{3}{4}$ as many days as A, and C can do it in $\frac{2}{3}$ as many days as B; together they can do the work in $3\frac{3}{23}$ days. In how many days can each alone do the work?

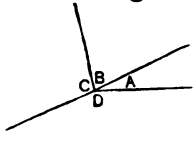
31. Two persons, A and B, 63 miles apart, start at the same time and travel toward each other. A travels at the rate of 4 miles an hour, and B at the rate of 3 miles an hour. How far will each have travelled when they meet?

It is often advantageous to represent the unknown number by some *multiple* of a letter.

Then let $4x$ = the number of miles that A travels,
and $3x$ = the number of miles that B travels.

32. A man started from his home to catch a train at the rate of one yard in a second, and arrived 2 minutes late. If he had walked at the rate of 4 yards in 3 seconds, he would have been $3\frac{1}{2}$ minutes too early. Find the distance to the station.

33. From a point O are drawn four lines forming the angles A, B, C, and D. The sum of these angles is 360°, and they are in the proportion of 2, 3, 4, and 6. Find each angle.



34. Find the mean proportional between

$$\frac{x^2-x-12}{x-5} \text{ and } \frac{x^2-9x+20}{x+3}.$$

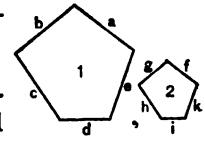
35. Use § 147, then solve for m:

$$\frac{3m-8}{3m+5}=\frac{2m-6}{2m+7}.$$

36. The numerator of a fraction exceeds the denominator by 5. If the numerator be decreased by 9, and the denominator increased by 6, the sum of the resulting fraction and the given fraction is 2. Find the fraction.

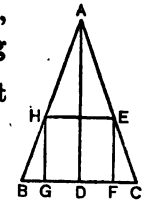
37. The areas of two similar pentagons (figures having five sides) are proportional to the squares of any two cor-

responding sides. The area of the first pentagon is 240, and the side a is 10; the area of the second pentagon is 60. Find the side f.



38. The width of a field is $\frac{2}{3}$ its length. If the width were increased by 5 feet, and the length by 10 feet, the area would be increased by 400 square feet. Find the dimensions.

39. A rectangle, $GHEF$, whose base, GF, is 6, is inscribed in an isosceles triangle, ABC, whose altitude, AD, is 14, and whose base, BC, is 10. Knowing from geometry that $\frac{AD}{DC}=\frac{EF}{FC}$, $BD=DC$, and that $GD=DF$, find EF.



40. $.2\,x+.001-.03\,x=.113\,x-.0161$.

Transposing, $.2\,x-.03\,x-.113\,x=-.0161-.001$.

Uniting terms, $.057\,x=-.0171$.

Dividing by .057, $x=-.3$.

41. $7.98\,x-3.75=.23\,x+.125$.

42. $3\,t+.052-7.8\,t=.04-5.82\,t-.0696$.

43. $.05\,v-1.82-.7\,v=.008\,v-.504$.

44. $.73\,D+8.86=.6(2.3\,D-.4)$.

45. $.07(8\,s-5.7)=.8(5\,s+.86)+1.321$.

*46. The density of a substance is defined as the number of grams in one cubic centimeter. Hence the total number of grams, M, in any body is equal to its density, D, multiplied by its volume, V; or, to state this relation algebraically,

$$M=DV,$$

V being given in cubic centimeters, and D in grams.

Two blocks, one of iron and one of copper, weigh the same number of grams; the iron has a volume of 10 cubic centi-

* Note: The metric tables will be found on page 228.

meters and a density of 7.4; the copper has a density of 8.9. Find the volume of the copper block.

47. When 100 grams of alcohol, of density .8, is poured into a cylindrical vessel, it is found to fill it to a depth of 10 centimeters. Find the area of the base of the cylinder in square centimeters.

48. A cylindrical iron bar, 2 centimeters in diameter, has a mass of 3 kilograms. Find the length of the bar.

Let $\pi = 3\frac{1}{7}$.

49. When a body is weighed under water, it is found to be buoyed up by a force equal to the weight of the water which it displaces.

If a boy can exert a lifting force of 120 pounds, how heavy a stone can he lift to the surface of a pond, if the density of stone is 2.5 and that of water 1 ?

50. When a straight bar is supported at some point, o, and masses m_1, m_2, etc., are hung from the bar as indicated in the figure, it is found that when the bar is in equilibrium, the following relation always holds,

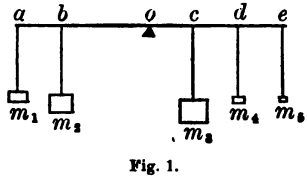


Fig. 1.

$$m_1 \cdot ao + m_2 \cdot bo = m_3 \cdot co + m_4 \cdot do + m_5 \cdot eo.$$

If a teeter board is 10 feet long, where must the support be placed in order that a 70-pound boy at one end may balance a 60-pound boy on the other end plus a 40-pound boy 3 feet from the other end?

51. A bar 40 inches long is in equilibrium when weights of 6 pounds and 9 pounds hang from its two ends. Find the position of the support.

52. If in Fig. 1, $ao = 100$, $bo = 40$, $co = 30$, $do = 60$, $eo = 110$, and if $m_1 = 40$, $m_2 = 60$, $m_3 = 60$, $m_4 = 15$, and $m_5 = 5$, where

must a mass of 100 be placed in order to produce equilibrium?

53. A gas expands $\frac{1}{273}$ of its volume at 0° centigrade for each degree of rise in its temperature; *i. e.*, the volume V_t, at any temperature, t, is connected with the volume V_0, at the temperature 0° centigrade by the equation

$$V_t = V_0 + \tfrac{1}{273} V_0 t,$$

or

$$V_t = V_0(1 + \tfrac{1}{273} t).$$

To what volume will 100 cubic centimeters of air at 0° expand when the temperature rises to 50° centigrade?

54. To what volume will 100 cubic centimeters of air at 50° centigrade contract when the temperature falls to 0° centigrade?

55. To what volume will 100 cubic centimeters of air at 50° expand when the temperature changes to 75°?

56. When a body in motion collides with a body at rest, the momentum of the first body (*i. e.*, the product of its mass, m_1, by its original velocity, v_1) is found to be in every case exactly equal to the total momentum of the two bodies after collision (*i. e.*, to the product of the mass, m_2, of the second body times the velocity, v_2, which it acquires, plus the product of m_1 by the velocity, v_3, which it retains after the collision). The algebraic statement of this relation is

$$m_1 v_1 = m_2 v_2 + m_1 v_3.$$

A billiard ball, the mass of which is 50 grams, and which was moving at a velocity of 1500 centimeters a second, collided with another ball at rest which weighed 30 grams. In the collision the first ball imparted to the second a velocity of 1600 centimeters per second. Find the velocity of the first ball after the collision.

PROBLEMS INVOLVING LITERAL EQUATIONS

150. *Prob.* Divide a into two parts such that m times the first shall exceed n times the second by b.

Let x = one part.

Then, $a-x$ = the other part.

By the conditions, $mx=n(a-x)+b$.

$$mx=an-nx+b,$$

$$mx+nx=an+b,$$

$$x(m+n)=an+b.$$

Whence, $$x=\frac{an+b}{m+n}, \text{ the first part.} \qquad (1)$$

And, $$a-x=a-\frac{an+b}{m+n}=\frac{am+an-an-b}{m+n}$$

$$=\frac{am-b}{m+n}, \text{ the other part.} \qquad (2)$$

The results can be used as FORMULÆ *for solving any problem of the above form.*

Thus, let it be required to divide 25 into two parts such that 4 times the first shall exceed 3 times the second by 37.

Here, $a=25$, $m=4$, $n=3$, and $b=37$.

Substituting these values in (1) and (2),

the first part $$=\frac{25\times3+37}{7}=\frac{75+37}{7}=\frac{112}{7}=16,$$

and the second part $$=\frac{25\times4-37}{7}=\frac{100-37}{7}=\frac{63}{7}=9.$$

EXERCISE 60

1. Divide a into two parts whose quotient shall be m.

2. If A can do a piece of work in m hours, and A and B together in n hours, in how many hours can B alone do the work?

3. Divide a into two parts such that the sum of one-mth the first and one-nth the second shall equal b.

4. A courier who travels a miles a day is followed by another who travels b miles a day. How many days must the second start after the first to overtake him after c days?

5. Divide a into three parts such that the first shall be one-mth the second and one-nth the third.

6. The length of a field is m times its width. If the length were increased by a feet, and the width by b feet, the area would be increased by c square feet. Find the dimensions of the field.

7. A courier who travels a miles a day is followed after b days by another. How many miles a day must the second courier travel to overtake the first after c days?

8. If A can do a piece of work in a hours, B in b hours, C in c hours, and D in d hours, how many hours will it take to do the work if all work together?

XI. SIMULTANEOUS LINEAR EQUATIONS

CONTAINING TWO OR MORE UNKNOWN NUMBERS

151. An equation containing two or more unknown numbers is satisfied by an unlimited number of sets of values of these numbers.

Consider, for example, the equation $x+y=5$.
Putting $x=1$, we have $1+y=5$, or $y=4$.
Putting $x=2$, we have $2+y=5$, or $y=3$; etc.
Thus the equation is satisfied by the sets of values

$$x=1,\ y=4,$$

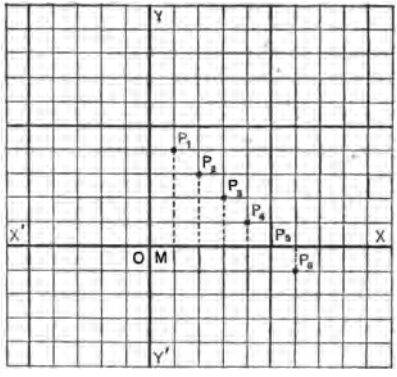
and

$$x=2,\ y=3;\ \text{etc.}$$

An algebraic equation which is satisfied by an unlimited number of such sets of values, is called an **Indeterminate Equation.**

If we agree, as in example 28, Exercise 41, that distances measured toward the right from a definite line and upward from another definite line shall be positive and that measurements in the opposite directions be negative, and also that the vertical measurements shall be y *measurements* and the horizontal distances x *measurements*, a definite picture of the equation $x+y=5$ may be drawn. On *square ruled* paper, choose a horizontal and a vertical line, $X'X$ and $Y'Y$; these lines are called the *x-axis* and *y-axis* respectively.

When $x=1, y=4$, laying off $OM=1$, and $MP_1=4$, we locate the point P_1. *1* and *4* are called the *coördinates* of the point P_1. MP_1 is the *ordinate* and OM the *abscissa* of the point P_1. O is the *origin*. Taking other pairs of values of x and y which satisfy $x+y=5$, we may locate the points P_2, P_3, etc., obtained from $x=2$, $y=3$; $x=3, y=2$; $x=4, y=1$; $x=5, y=0$; $x=6, y=-1$; etc. Connect these points.



The line thus drawn is called the *grah* of the given equation. The graph of an equation of the first degree in x and y (§ 75), is a straight line. Therefore the equation is called *linear*.

Coördinates are often written thus, (x, y), the x coördinate being written first.

EXERCISE 61

1. Locate the points $(2, 5)$; $(3, -2)$; $(-3, 2)$; $(-5, -1)$; $(2, 7)$; $(-9, 4)$.

2. Construct the graph of: $x-y=5$.

3. Construct the graph of: $2x+y=8$.

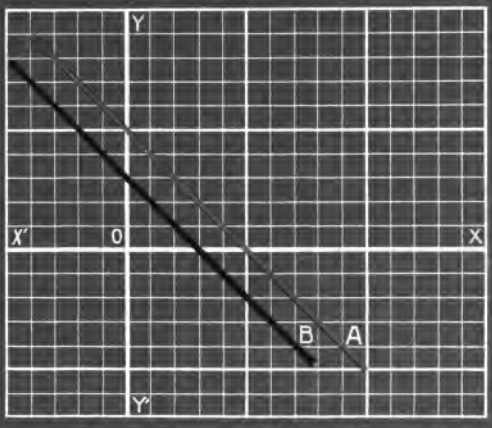
152. Consider the equations

$$\begin{cases} x+y=5, & (1) \\ 2x+2y=10. & (2) \end{cases}$$

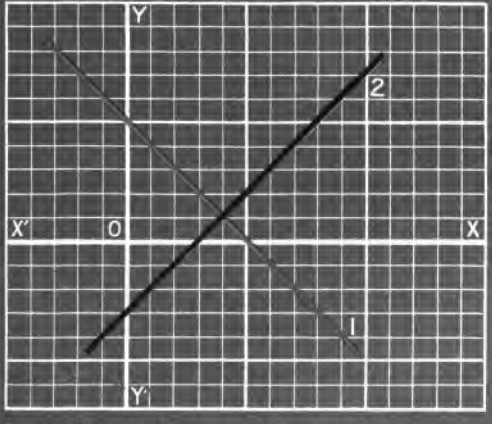
Equation (1) can be made to take the form of (2) by multiplying both members by 2; then, every set of values of x and y which satisfies one of the equations also satisfies the other. Such equations are called **Equivalent.**

Again, consider the equations

$$\begin{cases} x+y=5, & (3) \\ x-y=3. & (4) \end{cases}$$



Graphs of equations in § 153



Graphs of equations in § 154

PLATE I

In this case, it is not true that every set of values of x and y which satisfies one of the equations also satisfies the other; thus, equation (3) is satisfied by the set of values $x=3$, $y=2$, which does not satisfy (4).

If two equations, containing two or more unknown numbers, are not equivalent, they are called **Independent.**

153. Consider the equations

$$\begin{cases} x+y=5, & \text{(A)} \\ x+y=3. & \text{(B)} \end{cases}$$

It is evidently impossible to find a set of values of x and y which shall satisfy both (A) and (B). Such equations are called **Inconsistent.**

154. A system of equations is called **Simultaneous** when each contains two or more unknown numbers, and every equation of the system is satisfied by the same set, or sets, of values of the unknown numbers; thus, each equation of the system

$$\begin{cases} x+y=5, & (1) \\ x-y=3, & (2) \end{cases}$$

is satisfied by the set of values $x=4$, $y=1$.

In § 151, we found that one equation containing two unknown quantities was *indeterminate*. Notice that the *graphs*, Plate I, of $x+y=5$ (1) and $x-y=3$ (2) do not have the same direction. If constructed on the same diagram these lines will cross at some point. Construct the graphs on the same diagram, *i. e.*, use the same axes for both equations, and you will find that the coördinates of the crossing point are the same as the x and y of the set of values in § 154, *i. e.*, $x=4$, $y=1$. The coördinates of every point on the graph of equation (1) satisfy equation (1), also the coördinates of every point on the graph of equation (2) satisfy equation (2), but only at this point (4, 1) do the same coördinates satisfy both equations, hence the name *simultaneous equations*. In solving simultaneous equations we are simply finding the point where the graphs intersect. The finding of this point by means of graphs is somewhat slow and inaccurate, but *algebra* offers several methods by which the point may be readily found.

A Solution of a system of simultaneous equations is a set of values of the unknown numbers which satisfies every equa-

tion of the system; to *solve* a system of simultaneous equations is to find its solutions.

155. Two independent simultaneous equations of the form $ax+by=c$ may be solved by combining them in such a way as to form a single equation containing but *one* unknown number. This operation is called Elimination.

ELIMINATION BY ADDITION OR SUBTRACTION

156. 1. Solve the equations $\begin{cases} 5x-3y=19. & (1) \\ 7x+4y=2. & (2) \end{cases}$

Multiplying (1) by 4,	$20x-12y=76.$	(3)
Multiplying (2) by 3,	$21x+12y=6.$	(4)
Adding (3) and (4),	$41x=82.$	(5)
Whence,	$x=2.$	(6)
Substituting $x=2$ in (1),	$10-3y=19.$	(7)
Whence,	$-3y=9$, or $y=-3.$	(8)

Check this solution by substituting $x=2$, $y=-3$ in the *given* equations.

The above is an example of elimination by addition.

We speak of *adding* a system of equations when we mean placing the sum of the first members equal to the sum of the second members.

Abbreviations of this kind are frequent in Algebra; thus we speak of *multiplying* an equation when we mean multiplying each term of both of its members.

2. Solve the equations $\begin{cases} 15x+8y=1. & (1) \\ 10x-7y=-24. & (2) \end{cases}$

Multiplying (1) by 2,	$30x+16y=2.$	(3)
Multiplying (2) by 3,	$30x-21y=-72.$	(4)
Subtracting (4) from (3),	$37y=74$, and $y=2.$	
Substituting $y=2$ in (1),	$15x+16=1.$	
Whence,	$15x=-15$, and $x=-1.$	

The above is an example of elimination by *subtraction*.

From the above examples, we have the following rule:

If necessary, multiply the given equations by such numbers as will make the coefficients of one of the unknown numbers in the resulting equations of equal absolute value.

Add or subtract the resulting equations according as the coefficients of equal absolute value are of unlike or like sign.

If the coefficients which are to be made of equal absolute value are prime to each other, each may be used as the multiplier for the other equation; but if they are not prime to each other, such multipliers should be used as will produce their lowest common multiple.

Thus, in Ex. 1, to make the coefficients of y of equal absolute value, we multiply (1) by 4 and (2) by 3; but in Ex. 2, to make the coefficients of x of equal absolute value, since the L. C. M. of 10 and 15 is 30, we multiply (1) by 2 and (2) by 3.

EXERCISE 62

Solve by the method of addition or subtraction; verify each result:

1. $\begin{cases} 6x+5y=28. \\ 4x+\ y=14. \end{cases}$

2. $\begin{cases} x-5y=-21. \\ 3x-8y=-35. \end{cases}$

3. $\begin{cases} 2t-3u=19. \\ 7t+4u=23. \end{cases}$

4. $\begin{cases} 11x-15y=-\ 7. \\ 5y+\ 9x=-23. \end{cases}$

5. $\begin{cases} 3u-2v=8. \\ 4u+3v=5. \end{cases}$

6. $\begin{cases} 8m+6k=9. \\ 12m-9k=8. \end{cases}$

ELIMINATION BY SUBSTITUTION

157. *Ex.* Solve the equations $\begin{cases} 7x-9y=15. & (1) \\ 8y-5x=-17. & (2) \end{cases}$

Transposing $-5x$ in (2), $8y=5x-17.$

Whence, $y=\dfrac{5x-17}{8}.$ (3)

Substituting in (1), $7x-9\left(\dfrac{5x-17}{8}\right)=15.$ (4)

Clearing of fractions, $56x-9(5x-17)=120.$

Or, $56x-45x+153=120.$

Uniting terms, $11x=-33.$

Whence, $x=-3.$ (5)

Substituting $x=-3$ in (3), $y=\dfrac{-15-17}{8}=-4.$ (6)

Verify the result.

From the above example, we have the following rule:

From one of the given equations find the value of one of the unknown numbers in terms of the other, and substitute this value in place of that number in the other equation.

ELIMINATION BY COMPARISON

158. *Ex.* Solve the equations $\begin{cases} 2x-5y=-16. & (1) \\ 3x+7y=5. & (2) \end{cases}$

Transposing $-5y$ in (1), $2x=5y-16.$

Whence, $x=\frac{5y-16}{2}.$ (3)

Transposing $7y$ in (2), $3x=5-7y.$

Whence, $x=\frac{5-7y}{3}.$ (4)

Equating values of x, $\frac{5y-16}{2}=\frac{5-7y}{3}.$ (5)

Clearing of fractions, $15y-48=10-14y.$

Transposing, $29y=58.$

Whence, $y=2.$ (6)

Substituting $y=2$ in (3), $x=\frac{10-16}{2}=-3.$ (7)

Verify your result.

From the above example, we have the following rule:

From each of the given equations, find the value of the same unknown number in terms of the other, and place these values equal to each other.

EXERCISE 63

Solve by either substitution or comparison, using the method that seems the easier. Verify each result.

1. $\begin{cases} 3v-5y=-5. \\ 7v-3y=10. \end{cases}$

2. $\begin{cases} x+2y=9. \\ x-9y=7. \end{cases}$

3. $\begin{cases} 4w-l=11. \\ w+l=12. \end{cases}$

4. $\begin{cases} 15t+8u=3. \\ 6t-12u=5. \end{cases}$

5. $\begin{cases} 6z+11y=31. \\ 6y-11z=74. \end{cases}$

6. $\begin{cases} 5q+2r=-4. \\ 6q-11r=-45. \end{cases}$

7. $\begin{cases} 12e-11f=19. \\ 12f-11e=-27. \end{cases}$

8. $\begin{cases} 9u+6v=-16. \\ 13u+7v=-22. \end{cases}$

9. $\begin{cases} 8m-15v=18. \\ 12m+6v=-11. \end{cases}$

10. $\begin{cases} 5h+4k=34. \\ 8h-3k=35. \end{cases}$

EXERCISE 64

Solve the following equations, using any method of elimination you choose, and verify each result:

1. $\begin{cases} x+2y=-2. \\ 4x-7y=37. \end{cases}$

2. $\begin{cases} 11v+4u=3. \\ 8v+9u=-10. \end{cases}$

3. $\begin{cases} 6t-7w=-12. \\ 10t-9w=-12. \end{cases}$

4. $\begin{cases} 15r+8s=-14. \\ 6r+12s=1. \end{cases}$

5. $\begin{cases} \frac{2x}{3}+\frac{3y}{4}=-\frac{7}{2}. \\ \frac{x}{4}-\frac{2y}{5}=\frac{11}{2}. \end{cases}$

If the given equations are not in the form $ax+by=c$, they should be reduced to this form before applying any method of elimination.

6. $\begin{cases} 8e+7f=12. \\ \frac{e+2f}{4}+\frac{2e+f}{3}=1. \end{cases}$

7. $\begin{cases} \frac{y}{2}-\frac{x}{2}=2. \\ \frac{3-2x}{5}-\frac{4+5y}{11}=4. \end{cases}$

8. $\begin{cases} \frac{x+2}{7}+8=2x-\frac{y-x}{4}. \\ 3x-\frac{2y-3x}{3}=2y-4. \end{cases}$

9. $\begin{cases} \frac{8-y}{5}-\frac{2u+3}{4}=\frac{y+3}{4}. \\ \frac{1+4y}{11}-\frac{u+7}{3}=3. \end{cases}$

In solving fractional simultaneous equations, we reject any solution which does not satisfy the given equations.

10. **Solve the equations** $\begin{cases} 2x+3y=13. & (1) \\ \frac{1}{x-2}+\frac{1}{y-3}=0. & (2) \end{cases}$

Multiplying each term of (2) by $(x-2)(y-3)$, we have

$$y-3+x-2=0, \text{ or } y=-x+5. \qquad (3)$$

Substituting in (1), $2x-3x+15=13$, or $x=2$.

Substituting in (3), $y=-2+5=3$.

This solution satisfies the first given equation, but not the second; then it must be rejected.

11. $\begin{cases} 3x-4y=-11. \\ \dfrac{2}{x+5}-\dfrac{5}{y+1}=0. \end{cases}$

12. $\begin{cases} x-\dfrac{4y-9}{11}=5. \\ \dfrac{9}{2}-\dfrac{x+5}{3}=-3y. \end{cases}$

13. $\begin{cases} (2m-1)(z-4)-(m-5)(2z+5)=121. \\ 4m-3z=-29. \end{cases}$

14. $\begin{cases} \dfrac{7}{u-3}-\dfrac{8}{v-5}=0. \\ \dfrac{9}{2u-1}-\dfrac{5}{3v+4}=0. \end{cases}$

15. $\begin{cases} \dfrac{x+11}{7}+\dfrac{y-6}{5}=-4. \\ \dfrac{x-1}{2}-\dfrac{y+4}{10}=-45. \end{cases}$

16. $\begin{cases} \dfrac{d-2n}{3d+n+3}=-\dfrac{1}{5}. \\ \dfrac{d+3n}{d+4n-7}=\dfrac{7}{11}. \end{cases}$

17. $\begin{cases} .08x+.9y=.048. \\ .3x-.35y=.478. \end{cases}$

18. $\begin{cases} \dfrac{2x-3y}{4}+\dfrac{4x+6y}{3}=-\dfrac{1}{2}. \\ \dfrac{5x+2y}{2}+\dfrac{7y-3x}{5}=\dfrac{39}{10}. \end{cases}$

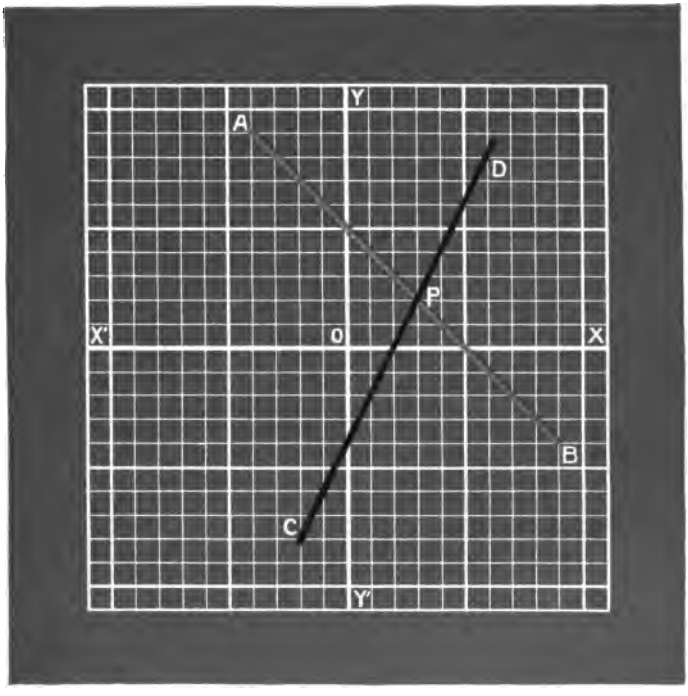
19. $\begin{cases} \dfrac{h+k}{h-k}=-\dfrac{1}{10}. \\ \dfrac{3h+8}{k-4}=\dfrac{6h-1}{2k+3}. \end{cases}$

20. $\begin{cases} 5x-\frac{1}{8}(3x-2y+5)=11. \\ \frac{5}{6}(x-4y)-\frac{4}{9}(x-y)=16. \end{cases}$

21. $\begin{cases} \dfrac{21g+1}{2}-\dfrac{5v+2}{3}=\dfrac{63g-130v}{21}. \\ \dfrac{14g+1}{3}-\dfrac{10v-3}{5}=2. \end{cases}$

Certain equations in which the unknown numbers occur in the denominators of fractions may be readily solved without previously clearing of fractions.

22. $\begin{cases} \dfrac{10}{x}-\dfrac{9}{y}=8. & (1) \\ \dfrac{8}{x}+\dfrac{15}{y}=-1. & (2) \end{cases}$



$x+y=5$ (AB)

$2x-y=4$ (CD)

$x+y=5$	
x	y
0	5
1	4
2	3
3	2
4	1
+5	0
−1	6

$2x-y=4$	
x	y
0	−4
1	−2
2	0
3	2
4	4
−1	−6
−2	−8

Solving the equations for x and y, we have $x=3$, $y=2$. Note that these values correspond to the x and y coördinates of the point P.

In general, in solving two simultaneous equations in two unknown quantities, the real values found for the unknown quantities correspond to the intersection points of the graphs of the given equations.

PLATE II

Multiplying (1) by 5, $\frac{50}{x}-\frac{45}{y}=40.$

Multiplying (2) by 3, $\frac{24}{x}+\frac{45}{y}=-3.$

Adding, $\frac{74}{x}=37,\ 74=37x,$ and $x=2.$

Substituting in (1), $5-\frac{9}{y}=8,\ -\frac{9}{y}=3,$ and $y=-3.$

23. $\begin{cases}\frac{3}{x}-\frac{3}{y}=-\frac{1}{4}.\\ \frac{6}{x}-\frac{3}{y}=\frac{1}{2}.\end{cases}$

24. $\begin{cases}\frac{6}{t}+\frac{12}{w}=-1.\\ \frac{8}{t}-\frac{9}{w}=7.\end{cases}$

25. $\begin{cases}\frac{3}{u}-\frac{4}{v}=1.\\ \frac{4}{u}+\frac{3}{v}=2.\end{cases}$

26. $\begin{cases}\frac{9}{x}-3y=4.\\ \frac{3}{x}+2y=\frac{10}{3}.\end{cases}$

27. $\begin{cases}\frac{6}{r}-\frac{2}{s}=-7.\\ \frac{4}{r}+\frac{6}{s}=-1.\end{cases}$

28. $\begin{cases}3m-\frac{5}{k}=10.\\ 2m+\frac{10}{k}=4.\end{cases}$

159. In graphical work the drawings are much more effective and pleasing if one uses a color scheme similar to that in Plate II. Cross-ruled paper and color crayons, for either blackboard or paper, can be secured at nominal expense.

EXERCISE 65

Construct the graphs of the following equations, in each case comparing the coördinates of the intersection with your algebraic solution:

1. $\begin{cases}x+y=8.\\ x-y=6.\end{cases}$

2. $\begin{cases}2x+y=10.\\ x+2y=4.\end{cases}$

3. $\begin{cases}5x-6y=-9.\\ 3x-5y=-4.\end{cases}$

4. $\begin{cases}8x-3y=47.\\ 6x-7y=21.\end{cases}$

5. $\begin{cases}x+2y=10.\\ 2x+4y=30.\end{cases}$

6. $\begin{cases}2x-3y=12.\\ 4x-6y=24.\end{cases}$

7. $\begin{cases} 3x-4y=10. \\ 4x+3y=5. \end{cases}$ 8. $\begin{cases} x-y=5. \\ x-y=8. \end{cases}$

9. $x+y=8$. Find the area of the triangle formed by this line and the axes.

10. $\begin{cases} 2x-y=2. \\ 4x-2y=16. \end{cases}$ 11. $\begin{cases} 3x-y=-9. \\ 3x+2y=-6. \end{cases}$

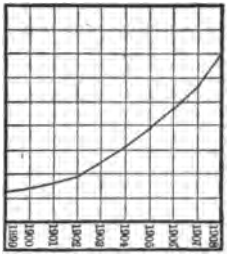
An interesting application of the graph is the construction of the geometric picture of related data:

12. The enrollment of pupils in the Seattle high school for ten years is as follows:

Year	No. of Pupils	Year	No. of Pupils	Year	No. of Pupils
1899	592	1903	1213	1906	2312
1900	684	1904	1522	1907	2794
1901	800	1905	1960	1908	3500
1902	947				

Choosing 1 year for the horizontal unit and 500 for the vertical unit, we have the following graph:

Vertical Scale, 500=1 Unit
Horizontal Scale, 1=1 Unit



Horizontal	*Vertical*
0	592
1	684
2	800
3	947
4	1213
5	1522
6	1960
7	2312
8	2794
9	3500

Construct the graphs of the following:

13. The enrollment in the Toledo high school is

Year	No. of Pupils	Year	No. of Pupils	Year	No. of Pupils
1898	773	1902	1376	1905	1622
1899	984	1903	1414	1906	1791
1900	1110	1904	1500	1907	1900
1901	1102				

14. Standing of the Chicago National League Baseball Team:

Year	Percentage	Year	Percentage	Year	Percentage
1898	.567	1902	.497	1905	.601
1899	.507	1903	.594	1906	.763
1900	.474	1904	.608	1907	.704
1901	.381				

15. Enrollment in Cleveland high schools:

Year	No. of Pupils	Year	No. of Pupils	Year	No. of Pupils
1898	3378	1901	3595	1904	4491
1899	3460	1902	3796	1905	5001
1900	3589	1903	4151	1906	5070

16. Enrollment in Chicago high schools:

Year	No. of Pupils	Year	No. of Pupils	Year	No. of Pupils
1897	7847	1901	9661	1904	9936
1898	8432	1902	9627	1905	11208
1899	8830	1903	9488	1906	12024
1900	9190				

17. Standing of the Detroit American League Baseball Team:

Year	Percentage	Year	Percentage	Year	Percentage
1901	.548	1904	.408	1906	.477
1902	.385	1905	.516	1907	.613
1903	.478				

18. Enrollment in New York City high schools:

Year	No. of Pupils	Year	No. of Pupils	Year	No. of Pupils
1899	13731	1902	21461	1905	30340
1900	17018	1903	23701	1906	31949
1901	19013	1904	27794		

160. Solution of Literal Simultaneous Equations. — In solving *literal* simultaneous linear equations, the method of elimination by addition or subtraction is usually to be preferred.

Ex. Solve the equations $\begin{cases} ax+by=c. & (1) \\ a'x+b'y=c'. & (2) \end{cases}$

Multiplying (1) by b', $ab'x+bb'y=b'c.$

Multiplying (2) by b, $a'bx+bb'y=bc'.$

Subtracting $(ab'-a'b)x=b'c-bc'.$

Whence, $x=\frac{b'c-bc'}{ab'-a'b}$.

Multiplying (1) by a', $aa'x+a'by=ca'.$ (3)

Multiplying (2) by a, $aa'x+ab'y=c'a.$ (4)

Subtracting (3) from (4), $(ab'-a'b)y=c'a-ca'.$

Whence, $y=\frac{c'a-ca'}{ab'-a'b}$.

In solving *fractional* literal simultaneous equations, any solution which does not satisfy the given equations must be rejected. (Compare Ex. 10, Exercise 64.)

EXERCISE 66

Solve the following:

1. $\begin{cases}5x-6y=8a.\\4x+9y=7a.\end{cases}$

2. $\begin{cases}ax+by=1.\\cx+dy=1.\end{cases}$

3. $\begin{cases}a_1x+a_2y=b_1.\\a_2x-a_1y=b_2.\end{cases}$

4. $\begin{cases}mx-ny=mn.\\m'x+n'y=m'n'.\end{cases}$

5. $\begin{cases}\frac{2ax-by}{a}=b.\\\frac{x+by}{3a+2}=b.\end{cases}$

6. $\begin{cases}\frac{x}{m_1}+\frac{y}{m_2}=\frac{1}{m_3}.\\\frac{x}{n_1}+\frac{y}{n_2}=\frac{1}{n_3}.\end{cases}$

7. $\begin{cases}bx-ay=b^2.\\(a-b)x+by=a^2.\end{cases}$

8. $\begin{cases}\frac{m}{n+y}=\frac{n}{m-x}.\\\frac{m}{n+x}=\frac{n}{m-y}.\end{cases}$

9. $\begin{cases}ax+by=2a.\\a^2x-b^2y=a^3+b^2.\end{cases}$

10. $\begin{cases}(a+1)x+(a-2)y=3a.\\(a+3)x+(a-4)y=7a.\end{cases}$

11. $\begin{cases}ab(a-b)x+ab(a+b)y=a^2+2ab-b^2.\\ax+by=2.\end{cases}$

12. $\begin{cases}\frac{a}{x}+\frac{b}{y}=c.\\\frac{a'}{x}+\frac{b'}{y}=c'.\end{cases}$

13. $\begin{cases}\frac{a}{bx}+\frac{b}{ay}=\frac{a+b}{ab}.\\\frac{b}{ax}-\frac{a}{by}=\frac{b^3-a^3}{a^2b^2}.\end{cases}$

14. $\begin{cases}m(x+y)+n(x-y)=2.\\m^2(x+y)-n^2(x-y)=m-n.\end{cases}$

15. $\begin{cases}(a+b)x+(a-b)y=2(a^2+b^2).\\\frac{b}{x-a-b}=\frac{a}{y-a+b}.\end{cases}$

16. $\begin{cases} (a+b)x+(a-b)y=2a^2-2b^2. \\ \dfrac{y}{a-b}-\dfrac{x}{a+b}=\dfrac{4ab}{a^2-b^2}. \end{cases}$

17. $\begin{cases} bx+ay=2. \\ ab(a+b)x-ab(a-b)y=a^2+b^2. \end{cases}$

SIMULTANEOUS EQUATIONS CONTAINING MORE THAN TWO UNKNOWN NUMBERS

161. If we have *three* independent simultaneous equations, containing *three* unknown numbers, we may combine any two of them by one of the methods of elimination explained in §§ 156 to 158, so as to obtain a single equation containing only two unknown numbers.

We may then combine the remaining equation with either of the other two, and obtain another equation containing the *same* two unknown numbers.

By solving the two equations containing two unknown numbers, we may obtain their values; and substituting them in either of the given equations, the value of the remaining unknown number may be found.

We proceed in a similar manner when the number of equations and of unknown numbers is greater than three.

The method of elimination by addition or subtraction is usually the most convenient.

In solving *fractional* simultaneous equations, any solution which does not satisfy the given equations must be rejected. (Ex. 10, Exercise 64.)

1. Solve the equations $\begin{cases} 6x-4y-7z=17. & (1) \\ 9x-7y-16z=29. & (2) \\ 10x-5y-3z=23. & (3) \end{cases}$

Multiplying (1) by 3,	$18x-12y-21z=51.$	
Multiplying (2) by 2,	$18x-14y-32z=58.$	
Subtracting,	$2y+11z=-7.$	(4)
Multiplying (1) by 5,	$30x-20y-35z=85.$	(5)
Multiplying (3) by 3,	$30x-15y-9z=69.$	(6)
Subtracting (5) from (6)	$5y+26z=-16.$	(7)

Multiplying (4) by 5, $10y+55z=-35.$
Multiplying (7) by 2, $10y+52z=-32.$
Subtracting, $3z=-3$, or $z=-1.$
Substituting in (7), $2y-11=-7$, or $y=2.$
Substituting in (1), $6x-8+7=17$, or $x=3.$

EXERCISE 67

Solve the following:

*1. $\begin{cases} 4x-3y=-5. \\ 4y-3z=-13. \\ 4z-3x=18. \end{cases}$

2. $\begin{cases} 4x-5y-6z=0. \\ x-y+z=1. \\ 9x+z=8. \end{cases}$

3. $\begin{cases} 3x+y-z=14. \\ x+3y-z=16. \\ x+y-3z=-10. \end{cases}$

4. $\begin{cases} g+h-k=24. \\ 4g+3h-k=61. \\ 6g-5h-k=11. \end{cases}$

5. $\begin{cases} 3x+5y=5. \\ 9x+5z=55. \\ 9y+3z=-30. \end{cases}$

6. $\begin{cases} 5m-y+4v=-5. \\ 3m+5y+6v=-20. \\ m+3y-8v=-27. \end{cases}$

7. $\begin{cases} 2x-5y=-26. \\ 7x+6z=-33. \\ \dfrac{3}{y-4}=\dfrac{4}{z+2}. \end{cases}$

8. $\begin{cases} 2x+4y-z=-2. \\ 18x-8y+4z=-25. \\ 10x+4y-9z=-30. \end{cases}$

9. $\begin{cases} 3p+4q+5r=10. \\ 4p-5q-3r=25. \\ 5p-3q-4r=21. \end{cases}$

10. $\begin{cases} 4u-11v-5w=9. \\ 8u+4v-w=11. \\ 16u+7v+6w=64. \end{cases}$

11. $\begin{cases} 8x+4y+3z=-52. \\ 5x-y+12z=-52. \\ 9x+7y-6z=-36. \end{cases}$

12. $\begin{cases} 6r-s+3t=42. \\ 10r-5s-t=2. \\ 6r-17s+4t=-46. \end{cases}$

13. $\begin{cases} 2x+5y+3z=-7. \\ 2y-4z=2-3x. \\ 5x+9y=5+7z. \end{cases}$

14. $\begin{cases} \dfrac{5}{x}-\dfrac{8}{y}=-3. \\ \dfrac{8}{y}-\dfrac{3}{z}=1. \\ \dfrac{25}{z}+\dfrac{7}{3x}=2. \end{cases}$

* Eliminate y from (1) and (2) you then have two equations in x and z; or add the three given equations.

15. $\begin{cases} \frac{2}{3x}+\frac{1}{y}=-\frac{3}{10}. \\ \frac{3}{4y}-\frac{1}{z}=\frac{7}{30}. \\ \frac{4}{5z}+\frac{1}{x}=\frac{1}{12}. \end{cases}$ 16. $\begin{cases} ax+by=\frac{a^3+b^3}{abc}. \\ by+cz=\frac{b^3+c^3}{abc}. \\ cz+ax=\frac{c^3+a^3}{abc}. \end{cases}$

PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS WITH TWO OR MORE UNKNOWN NUMBERS

162. In solving problems where two or more letters are used to represent unknown numbers, we must obtain from the conditions of the problem *as many independent equations* (§ 152) *as there are unknown numbers to be determined.*

1. Divide 81 into two parts such that three-fifths the greater shall exceed five-ninths the less by 7.

Let x = the greater part,
and y = the less.
By the conditions, $x+y=81$, (1)
and $\frac{3x}{5}=\frac{5y}{9}+7$. (2)
Solving (1) and (2), $x=45$, $y=36$.

2. If 3 be added to both numerator and denominator of a fraction, its value is $\frac{2}{3}$; and if 2 be subtracted from both numerator and denominator, its value is $\frac{1}{2}$; find the fraction.

Let n = the numerator,
and d = the denominator.
By the conditions, $\frac{n+3}{d+3}=\frac{2}{3}$,
and $\frac{n-2}{d-2}=\frac{1}{2}$.
Solving these equations, $n=7$, $d=12$; then, the fraction is $\frac{7}{12}$.

3. A sum of money was divided equally between a certain number of persons. Had there been 3 more, each would have received \$1 less; had there been 6 fewer, each would have received \$5 more. How many persons were there, and how much did each receive?

Let x = the number of persons,
and y = the number of dollars received by each.
Then, xy = the number of dollars divided.

Since the sum of money could be divided between $x+3$ persons, each of whom would receive $y-1$ dollars, and between $x-6$ persons, each of whom would receive $y+5$ dollars, $(x+3)(y-1)$ and $(x-6)(y+5)$ also represent the number of dollars divided.

Then, $(x+3)(y-1)=xy$,
and $(x-6)(y+5)=xy$.
Solving these equations, $x=12$, $y=5$.

4. The sum of the three digits of a number is 13. If the number, decreased by 8, be divided by the sum of its second and third digits, the quotient is 25; and if 99 be added to the number, the digits will be inverted. Find the number.

Let x = the first digit,
y = the second,
and z = the third.
Then, $100x+10y+z$ = the number,
and $100z+10y+x$ = the number with its digits inverted.
By the conditions of the problem,

$$x+y+z=13,$$

$$\frac{100x+10y+z-8}{y+z}=25,$$

and $$100x+10y+z+99=100z+10y+x.$$

Solving these equations, $x=2$, $y=8$, $z=3$; and the number is 283.

5. A crew can row 10 miles in 50 minutes down stream, and 12 miles in $1\frac{1}{2}$ hours against the stream. Find the rate in miles per hour of the current, and of the crew in still water.

Let x = number of miles an hour of the crew in still water,
and y = number of miles an hour of the current.
Then, $x+y$ = number of miles an hour of the crew down stream,
and $x-y$ = number of miles an hour of the crew up stream.

The number of miles an hour rowed by the crew is equal to the distance in miles divided by the time in hours.

Then, $$x+y=10\div\frac{5}{6}=12,$$

and $$x-y=12\div\frac{3}{2}=8.$$

Solving these equations, $x=10$, $y=2$.

6. A train running from A to B meets with an accident which causes its speed to be reduced to one-third of what it was before, and it is in consequence 5 hours late. If the accident had happened 60 miles nearer B, the train would have been only 1 hour late. Find the rate of the train before the accident, and the distance to B from the point of detention.

Let $3x$ = the number of miles an hour of the train before the accident.

Then, x = the number of miles an hour after the accident.

Let y = the number of miles to B from the point of detention.

The train would have done the last y miles of its journey in $\frac{y}{3x}$ hours; but owing to the accident, it does the distance in $\frac{y}{x}$ hours.

Then, $$\frac{y}{x}=\frac{y}{3x}+5. \qquad (1)$$

If the accident had occurred 60 miles nearer B, the distance to B from the point of detention would have been $y-60$ miles.

Had there been no accident, the train would have done this in $\frac{y-60}{3x}$ hours, and the accident would have made the time $\frac{y-60}{x}$ hours.

Then, $$\frac{y-60}{x}=\frac{y-60}{3x}+1. \qquad (2)$$

Subtracting (2) from (1), $\frac{60}{x}=\frac{60}{3x}+4$, or $\frac{40}{x}=4$; whence, $x=10$.

Then, the rate of the train before the accident was 30 miles an hour.

Substituting in (1), $\frac{y}{10}=\frac{y}{30}+5$, or $\frac{y}{15}=5$; whence, $y=75$.

EXERCISE 68

1. If the numerator of a fraction be decreased by 1, the value of the fraction is $\frac{1}{3}$, while if 7 be added to both numerator and denominator, the value of the fraction is $\frac{11}{16}$; find the fraction.

2. The sum of two numbers is 7. The ratio of their product to the product of three times the first number and the second increased by 2 is $\frac{1}{5}$. What are the numbers?

3. The sum of the two digits of a number is 14; and if 36 be added to the number, the digits will be inverted. Find the number.

4. Nine shares of N. Y. Central stock and 7 shares of Illinois Central stock cost $1702, and 5 shares of I. C. cost $35 more than 6 shares of N. Y. C. Find the cost of one share of each.

5. Find two numbers such that the ratio of the first number to itself increased by 3 is equal to the ratio of the second number to itself increased by $\frac{6}{5}$; and the sum of the two numbers is to twice their difference as 7 is to 4.

6. If 3 be added to the numerator of a fraction, and 7 subtracted from the denominator, its value is $\frac{6}{7}$; and if 1 be subtracted from the numerator, and 7 added to the denominator, its value is $\frac{2}{5}$. Find the fraction.

7. Find two numbers such that one shall be n times as much greater than a as the other is less than a; and the quotient of their sum by their difference equal to b.

8. A wheat field is 80 rods longer than it is wide, and the distance around the field is $1\frac{1}{2}$ miles. Find the length and breadth.

9. In plowing the long way of the above field, a farmer finds he can turn 33 twelve-inch furrows a day. At $3.50 per day for a man and team, what is the cost of plowing per acre?

10. C's age is three times the sum of A's and B's. Three times B's age added to A's is 12 years less than C's, and if 8 years be subtracted from C's age and this difference be divided by B's age, the quotient will be 4. Find their ages.

11. A rectangular mirror is 6 inches longer than it is wide. It is surrounded by a frame 3 inches wide, whose area is 216 square inches. How much wall space will the mirror and frame occupy?

12. A man had $3000 in a savings bank which paid him 3% interest. He drew out a part of his money and invested it in municipal bonds which paid him 5%. His annual

income from the entire sum was then \$126. Find the amount left in the savings account.

13. If we consider $y=kx$ a proportion, what is the ratio of y to x? Give k some definite value and make the graph of the equation. If perpendiculars are dropped from the graph to the x-axis, triangles are formed by the perpendiculars, the graph, and the x-axis. Are the triangles alike in form? Is the ratio of the altitude to the base the same in each? If k is given a different value from the one chosen, what effect does this have on the graph and on the triangles?

14. A rectangular field has the same area as another which is 6 rods longer and 2 rods narrower, and also the same area as a third which is 3 rods shorter and 2 rods wider. Find its dimensions.

15. Find three numbers such that the first with one-half the second and one-third the third shall equal 29; the second with one-third the first and one-fourth the third shall equal 28; and the third with one-half the first and one-third the second shall equal 36.

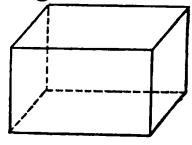
16. The circumference of the large wheel of a carriage is 55 inches more than that of the small wheel. The former makes as many revolutions in going 250 feet as the latter does in going 140 feet. Find the number of inches in the circumference of each wheel.

17. A man having \$4500 in a savings bank which paid him 3 % interest withdrew the money, investing a part in Rock Island 5 % bonds for which he paid \$80 (par value 100); with the balance he purchased Pennsylvania Railway 5 % bonds at par. His annual income from these investments was \$255. Find the amount invested in Rock Island bonds, and their face value.

18. A number consists of two digits. If the first digit be divided by one less than the second digit, the quotient is 3.

If the first digit, increased by 3, be divided by the second digit the quotient is 3; find the number. State your problem, then compare § 152.

19. The sum of the length, breadth, and height of a rectangular parallelopiped is 20. The difference between the length and height is $\frac{2}{5}$ the sum of the height and breadth, and three times the breadth added to the height is 8 more than the length; find the dimensions.



20. If the digits of a number of three figures be inverted, the sum of the number thus formed and the original number is 1615; the sum of the digits is 20, and if 99 be added to the number, the digits will be inverted. Find the number.

21. A train left A for B, 112 miles distant, at 9 A. M., and one hour later a train left B for A; they met at 12 noon. If the second train had started at 9 A. M., and the first at 9.50 A. M., they would also have met at noon. Find their rates.

22. A boy has $1.50 with which he wishes to buy two kinds of note-books. If he asks for 14 of the first kind, and 11 of the second, he will require 6 cents more; and if he asks for 11 of the first kind, and 14 of the second, he will have 6 cents over. How much does each kind cost?

23. The difference between the length and breadth of a rectangle is 6. If the length were diminished by 3 feet and the breadth increased by 3 feet, the area would be increased by 9 square feet and the figure would be a square; find the dimensions.

Have you more conditions than you need? Are your conditions independent? (Compare § 83.)

24. A number consisting of two digits is such that if the digits be inverted the number formed is 27 less than the original number. The product of the digits is to their difference as the second digit is to $\frac{8}{3}$; find the number.

25. A man invests \$10,000, part at $4\frac{1}{4}\%$, and the rest at $3\frac{1}{2}\%$. He finds that six years' interest on the first investment exceeds five years' interest on the second by \$658. How much does he invest at each rate?

26. A man buys apples, some at 2 for 3 cents, and others at 3 for 2 cents, spending in all 80 cents. If he had bought $\frac{5}{4}$ as many of the first kind, and $\frac{6}{5}$ as many of the second, he would have spent 99 cents. How many of each kind did he buy?

27. An annual income of \$800 is obtained in part from money invested at $3\frac{1}{2}\%$, and in part from money invested at 3%. If the amount invested at the first rate were invested at 3%, and the amount invested at the second rate were invested at $3\frac{1}{2}\%$, the annual income would be \$825. How much is invested at each rate?

28. The contents of one barrel is $\frac{5}{6}$ wine, and of another $\frac{3}{5}$ wine. How many gallons must be taken from each to fill a barrel whose capacity is 24 gallons, so that the mixture may be $\frac{7}{8}$ wine?

29. A boy spends his money for oranges. Had he bought m more, each would have cost a cents less; if n fewer, each would have cost b cents more. How many did he buy, and at what price?

30. A vessel contains a mixture of wine and water. If 50 gallons of wine are added, there is $\frac{7}{8}$ as much wine as water; if 50 gallons of water are added, there is 4 times as much water as wine. Find the number of gallons of wine and water at first.

31. A man buys 15 bottles of sherry, and 20 bottles of claret, for \$38. If the sherry had cost $\frac{5}{4}$ as much, and the claret $\frac{4}{5}$ as much, the wine would have cost \$38.50. Find the cost per bottle of the sherry, and of the claret.

32. If a field were made a feet longer, and b feet wider, its area would be increased by m square feet; but if its length

were made c feet less, and its width d feet less, its area would be decreased by n square feet. Find its dimensions.

33. If the numerator of a fraction be increased by a, and the denominator by b, the value of the fraction is $\frac{m}{n}$; and if the numerator be decreased by c, and the denominator by d, the value of the fraction is $\frac{n}{m}$. Find the numerator and denominator.

34. A certain number equals 59 times the sum of its three digits. The sum of the digits exceeds twice the ten's digit by 3; and the sum of the hundred's and ten's digits exceeds twice the unit's digit by 6. Find the number.

35. A piece of work can be done by A and B in $4\frac{4}{5}$ hours, by B and C in $2\frac{2}{3}$ hours, and by A and C in 3 hours. In how many hours can each alone do the work?

36. The numerator of a fraction has the same two digits as the denominator, but in reversed order; the denominator exceeds the numerator by 9, and if 1 be added to the numerator the value of the fraction is $\frac{3}{4}$. Find the fraction.

37. A man walks from one place to another in $5\frac{1}{2}$ hours. If he had walked $\frac{1}{4}$ of a mile an hour faster, the walk would have taken $36\frac{2}{3}$ fewer minutes. How many miles did he walk, and at what rate?

38. A man invests a certain sum of money at a certain rate of interest. If the principal had been \$1200 greater, and the rate 1% greater, his income would have been increased by \$118. If the principal had been \$3200 greater, and the rate 2% greater, his income would have been increased by \$312. What sum did he invest, and at what rate?

39. A crew row $16\frac{1}{2}$ miles up stream and 18 miles down stream in 9 hours. They then row 21 miles up stream and $19\frac{1}{2}$ miles down stream in 11 hours. Find the rate in miles an hour of the stream, and of the crew in still water.

40. A man buys a certain number of $100 railway shares, when at a certain rate per cent discount, for $1050; and when at a rate per cent premium twice as great, sells one-half of them for $1200. How many shares did he buy, and at what cost?

163. Interpretation of Solutions.

1. The length of a field is 10 rods, and its breadth 8 rods; how many rods must be added to the breadth so that the area may be 60 square rods?

Let x = number of rods to be added.

By the conditions, $10(8+x)=60$.

Then, $80+10x=60$, or $x=-2$.

This signifies that 2 rods must be *subtracted* from the breadth in order that the area may be 60 square rods. (Compare § 11.)

If we should modify the problem so as to read:

"The length of a field is 10 rods, and its breadth 8 rods; how many rods must be *subtracted* from the breadth so that the area may be 60 square rods?"

and let x denote the number of rods to be subtracted, we should find $x=2$.

A negative result sometimes indicates that the problem is impossible. It sometimes indicates that measurement is taken in an opposite direction (Ex. 28, Exercise 41).

2. If 11 times the number of persons in a certain house, increased by 18, be divided by 4, the result equals twice the number increased by 3; find the number.

Let x = the number.

By the conditions, $\frac{11x+18}{4}=2x+3$.

Whence, $11x+18=8x+12$, and $x=-2$.

The negative result shows that the problem is impossible.

A problem may also be impossible when the solution is fractional.

XII. INVOLUTION AND EVOLUTION

164. Involution is the process of raising an expression to any power whose exponent is a positive integer.

We gave in § 88 a rule for raising a monomial to any power whose exponent is a positive integer.

165. If an expression when raised to the nth power, n being a positive integer, is equal to another expression, the first expression is said to be the nth **Root** of the second.

Thus, if $a^n=b$, a is the nth root of b;
if $5^2=25$, 5 is the square root of 25.

Evolution is the process of finding any required root of an expression.

166. The Radical Sign, $\sqrt{\ }$, when written before an expression, indicates some root of the expression.

Thus, $\sqrt{a}$ indicates the *second*, or *square* root of a;
$\sqrt[3]{a}$ indicates the *third*, or *cube* root of a;
$\sqrt[4]{a}$ indicates the *fourth* root of a; and so on.

The **Index** of a root is the number written over the radical sign to indicate what root of the expression is taken.

If no index is expressed, the index 2 is understood.

An *even* root is one whose index is an even number; an *odd* root is one whose index is an odd number.

167. A Power of a Fraction.

We have, $\left(\frac{a}{b}\right)^3=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a\times a\times a}{b\times b\times b}=\frac{a^3}{b^3}$;

and a similar result holds for any positive integral power of $\frac{a}{b}$.

Then, a fraction may be raised to any power whose exponent is a positive integer by raising both numerator and denominator to the required power.

Ex. $\left(-\frac{2x^4}{3y^3}\right)^5 = -\left(\frac{2x^4}{3y^3}\right)^5$ (§ 49) $= -\frac{(2x^4)^5}{(3y^3)^5} = -\frac{32x^{20}}{243y^{15}}$ (§ 88).

EXERCISE 69

Find the values of the following:

1. $\left(\frac{6a^9b^6}{7c^8d^7}\right)^2$.
2. $\left(\frac{9mn^4}{8p^5}\right)^3$.
3. $\left(-\frac{3a^2x^7}{b^3y}\right)^5$.
4. $\left(-\frac{4x^m}{5y^4z^9}\right)^4$.
5. $\left(-\frac{2m^7x^2}{n^5y^4}\right)^7$.
6. $\left(\frac{a^3m^5}{3b^8cn^6}\right)^6$.

168. A Root of a Monomial. To find any root of a monomial which is a perfect power of the same degree as the index of the required root.

1. Required the cube root of $a^3b^9c^6$.

We have, $(ab^3c^2)^3 = a^3b^9c^6$.

Then, by § 165, $\sqrt[3]{a^3b^9c^6} = ab^3c^2$.

2. Required the fifth root of $-32a^5$.

We have, $(-2a)^5 = -32a^5$.

Whence, $\sqrt[5]{-32a^5} = -2a$.

Similarly, to extract a root of any monomial:

Extract the required root of the absolute value of the numerical coefficient, and divide the exponent of each letter by the index of the required root.

Give to every even root of a positive term the sign ±, and to every odd root of any term the sign of the term itself.

The sign ±, called the *double sign*, is prefixed to an expression when we wish to indicate that it is either + or −.

1. Find the square root of $9a^4b^6c^{10}$.

By the rule, $\sqrt{9a^4b^6c^{10}} = \pm 3a^2b^3c^5$.

It follows from §§ 167, 168, that, to find any root of a fraction, each of whose terms is a perfect power of the same degree as the index of the required root, *extract the required root of both numerator and denominator.*

2. $\sqrt[3]{-\frac{27\,a^3b^6}{64\,c^9}} = -\frac{\sqrt[3]{27\,a^3b^6}}{\sqrt[3]{64\,c^9}} = -\frac{3\,ab^2}{4\,c^3}$.

The root of a large number may sometimes be found by resolving it into its prime factors.

3. Find the square root of 254016.

We have, $\sqrt{254016} = \sqrt{2^6 \times 3^4 \times 7^2} = \pm 2^3 \times 3^2 \times 7 = \pm 504$.

4. Find the value of $\sqrt[3]{72 \times 75 \times 135}$.

$$\sqrt[3]{72 \times 75 \times 135} = \sqrt[3]{(2^3 \times 3^2) \times (3 \times 5^2) \times (3^3 \times 5)}$$
$$= \sqrt[3]{2^3 \times 3^6 \times 5^3} = 2 \times 3^2 \times 5 = 90.$$

EXERCISE 70

Find the values of the following:

1. $\sqrt{36\,x^2y^8}$.
2. $\sqrt[3]{64\,a^{12}b^3c^6}$.
3. $\sqrt[9]{-x^9y^{18}z^{27}}$.
4. $\sqrt{\frac{64\,x^2y^8}{49\,z^4}}$.
5. $\sqrt[3]{-\frac{27\,a^9}{125\,b^6}}$.
6. $\sqrt[4]{81\,n^{20}x^{12}y^4}$.
7. $\sqrt{121\,a^{12}b^{22}c^4}$.
8. $\sqrt[3]{-216\,x^{24}y^9z^{15}}$.
9. $\sqrt[7]{128\,m^{14}n^{21}}$.
10. $\sqrt[3]{-343\,x^{3m+9}y^{6n}}$.
11. $\sqrt[4]{625\,a^{16m}b^{4n}}$.
12. $\sqrt{\frac{m^{20}}{256\,n^8}}$.
13. $\sqrt[6]{\frac{a^{12m}}{729\,b^{6n}}}$.
14. $\sqrt{2916}$.
15. $\sqrt{30625}$.
16. $\sqrt{86436}$.
17. $\sqrt{25 \cdot 36 \cdot 196}$.
18. $\sqrt[3]{27 \cdot 64 \cdot 8}$.
19. $\sqrt{250 \cdot 32 \cdot 45}$.
20. $\sqrt[3]{4\,ab \cdot 144\,b^2c \cdot 24\,a^2c^2}$.
21. $\sqrt[3]{252\,a^3 \cdot 245\,d^3 \cdot 150\,c^3}$.
22. $\sqrt[5]{59049}$.
23. $\sqrt[3]{112 \cdot 168 \cdot 252}$.
24. $\sqrt{(a^2-5\,a+6)(a^2+2\,a-8)(a^2+a-12)}$.
25. $\sqrt{(2\,a^2+7\,a-15)(8\,a^2+2\,a-21)(4\,a^2+27\,a+35)}$.

169. It may be proved that $\sqrt[n]{(a^n)^m} = \sqrt[n]{a^{mn}} = a^m = (\sqrt[n]{a^n})^m$.

Ex. Required the value of $\sqrt[5]{(32\,a^{10})^4}$.

We have, $\sqrt[5]{(32\,a^{10})^4} = (\sqrt[5]{32\,a^{10}})^4 = (2\,a^2)^4 = 16\,a^8$.

This method of finding the root is shorter than raising $32\,a^{10}$ to the fourth power, and then taking the fifth root of the result.

EXERCISE 71

Find the values of the following:

1. $\sqrt[3]{(64\,a^3)^2}$.
2. $\sqrt{(4\,m^4)^7}$.
3. $\sqrt[4]{(16\,x^8y^{12})^5}$.
4. $\sqrt[5]{(-243\,a^5b^{25}c^{10})^3}$.
5. $\sqrt[6]{(64\,m^{12}n^6)^5}$.
6. $\sqrt[3]{\left(-\frac{8\,x^3}{27\,y^6}\right)^4}$.
7. $\sqrt{(a^2-2\,ab+b^2)^3}$.

170. Square of a Polynomial. — We find by actual multiplication:

$$\begin{array}{r} a+b+c \\ a+b+c \\ \hline a^2+ab+ac \\ +ab+b^2+bc \\ +ac+bc+c^2 \\ \hline a^2+2\,ab+2\,ac+b^2+2\,bc+c^2 \end{array}$$

The result, for convenience of enunciation, may be written:

$$(a+b+c)^2=a^2+b^2+c^2+2\,ab+2\,ac+2\,bc.$$

In like manner we find:

$$(a+b+c+d)^2=a^2+b^2+c^2+d^2 +2\,ab+2\,ac+2\,ad+2\,bc+2\,bd+2\,cd;$$

and so on.

We then have the following rule:

The square of a polynomial is equal to the sum of the squares of its terms, together with twice the product of each term by each of the following terms.

Ex. Expand $(2\,x^2-3\,x-5)^2$.

The squares of the terms are $4\,x^4$, $9\,x^2$, and 25.

Twice the product of the first term by each of the following terms gives the results $-12\,x^3$ and $-20\,x^2$.

Twice the product of the second term by the following term gives the result $30\,x$.

Then, $(2\,x^2-3\,x-5)^2=4\,x^4+9\,x^2+25-12\,x^3-20\,x^2+30\,x$
$=4\,x^4-12\,x^3-11\,x^2+30\,x+25.$

EXERCISE 72

Square each of the following:

1. $a+b+c$.
2. $3a-x+2y$.
3. $r+2s-3t$.
4. $a+d+3d^2$.
5. $2c-5a+m$.
6. $3x^2-2x-1$.
7. $2x^2+x-5$.
8. $4m^2-4m+1$.
9. $a^2+2ab+b^2$.
10. a^3+a^2-2a-2.

171. Square Root of a Polynomial by Inspection.— In § 91, we showed how to find the square root of a trinomial perfect square.

The square roots of certain polynomials of the form

$$a^2+b^2+c^2+2ab+2ac+2bc$$

can be found by inspection.

Ex. Find the square root of

$$9x^2+y^2+4z^2+6xy-12xz-4yz.$$

We can write the expression as follows:

$$(3x)^2+y^2+(-2z)^2+2(3x)y+2(3x)(-2z)+2y(-2z).$$

By § 170, this is the square of $3x+y+(-2z)$.

Then, the square root of the expression is $3x+y-2z$.

(The result could also have been obtained in the form $2z-y-3x$.)

EXERCISE 73

Find the square roots of the following:

1. $a^2+b^2+c^2-2ab-2ac+2bc$.
2. $x^2+4y^2+9+4xy+6x+12y$.
3. $1+25m^2+36n^2-10m+12n-60mn$.
4. $a^2+81b^2+16+18ab-8a-72b$.
5. $9x^2+y^2+25z^2-6xy-30xz+10yz$.
6. $36m^2+64n^2+x^2+96mn-12mx-16nx$.
7. $16a^4+9b^4+81c^4+24a^2b^2+72a^2c^2+54b^2c^2$.
8. $25x^6+49y^{10}+36z^8-70x^3y^5+60x^3z^4-84y^5z^4$.

172. Square Root of a Polynomial Perfect Square, general method.

By § 170, $(a+b+c)^2=a^2+2\,ab+b^2+2\,ac+2\,bc+c^2$

$$=a^2+(2\,a+b)b+(2\,a+2\,b+c)c. \quad (1)$$

Then, if the square of a trinomial be arranged in order of powers of some letter:

I. The square root of the first term gives the first term of the root, a.

II. If from (1) we subtract a^2, we have

$$(2\,a+b)b+(2\,a+2\,b+c)c. \quad (2)$$

The first term of this, when expanded, is $2\,ab$; if this be divided by twice the first term of the root, $2\,a$, we have the next term of the root, b.

III. If from (2) we subtract $(2\,a+b)b$, we have

$$(2\,a+2\,b+c)c. \quad (3)$$

The first term of this, when expanded, is $2\,ac$; if this be divided by twice the first term of the root, $2\,a$, we have the last term of the root, c.

IV. If from (3) we subtract $(2\,a+2\,b+c)c$, there is no remainder.

Similar considerations hold with respect to the square of a polynomial of any number of terms.

173. The principles of § 172 may be used to find the square root of a polynomial perfect square of any number of terms. Let it be required to find the square root of

$$4\,x^4+12\,x^3-7\,x^2-24\,x+16.$$

$$\begin{array}{rl|l}
 & 4\,x^4+12\,x^3-\ 7\,x^2-24\,x+16 & 2\,x^2+3\,x-4 \\
a^2=4\,x^4 & & \\
\hline
2\,a+b=4\,x^2+3\,x & 12\,x^3-\ 7\,x^2-24\,x+16, & \text{1st Rem.} \\
\quad 3\,x & 12\,x^3+\ 9\,x^2 & \\
\hline
2\,a+2\,b+c=4\,x^2+6\,x-4 & -16\,x^2-24\,x+16, & \text{2d Rem.} \\
\quad -4 & -16\,x^2-24\,x+16 &
\end{array}$$

The first term of the root is the square root of $4x^4$, or $2x^2$.

Subtracting the square of $2x^2$, $4x^4$, from the given expression, the first remainder is $12x^3-7x^2-24x+16$.

Dividing the first term of this by twice the first term of the root, $4x^2$, we have the next term of the root, $3x$ (§ 172, II).

Adding this to $4x^2$ gives $4x^2+3x$; multiplying the result by $3x$, and subtracting the product, $12x^3+9x^2$, from the first remainder, gives the second remainder, $-16x^2-24x+16$.

Dividing the first term of this by twice the first term of the root, $4x^2$, we have the last term of the root, -4 (§ 172, III).

If from the second remainder we subtract $(4x^2+6x-4)(-4)$, or $-16x^2-24x+16$, there is no remainder; then, $2x^2+3x-4$ is the required root (§ 172, IV).

The expressions $4x^2$ and $4x^2+6x$ are called *trial-divisors*, and $4x^2+3x$ and $4x^2+6x-4$ *complete divisors*.

We then have the following rule for extracting the square root of a polynomial perfect square:

Arrange the expression according to the powers of some letter.

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given expression, arranging the remainder in the same order of powers as the given expression.

Divide the first term of the remainder by twice the first term of the root, and add the quotient to the part of the root already found, and also to the trial-divisor.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

174. Examples.

1. Find the square root of $9x^4+30a^3x^2+25a^6$.

$$\begin{array}{r|l} 9x^4+30a^3x^2+25a^6 & 3x^2+5a^3 \\ \underline{9x^4} & \\ 6x^2+5a^3 \;\big|\; 30a^3x^2 & \\ \quad\big|\; \underline{30a^3x^2+25a^6} & \end{array}$$

It is usual, in practice, to omit those terms, after the first, in each remainder, which are merely repetitions of the terms in the given expres-

sion; thus, in the first remainder of Ex. 1, we leave out the term $25\,a^6$. It is also usual to leave out of the written work the multiplier of the complete divisor.

2. Find the square root of

$$20\,x^2-22\,x^3+1+28\,x^4+9\,x^6-8\,x-12\,x^5.$$

Arranging according to the descending powers of x, we have

$$\begin{array}{l|l}
9\,x^6-12\,x^5+28\,x^4-22\,x^3+20\,x^2-8\,x+1 & 3\,x^3-2\,x^2+4\,x-1 \\
\quad 9\,x^6 & \\
\hline
6\,x^3-2\,x^2 \;\big|\; -12\,x^5 & \\
\qquad\quad\;\; -12\,x^5+\;4\,x^4 & \\
\hline
6\,x^3-4\,x^2+4\,x \;\big|\; 24\,x^4 & \\
\qquad\qquad\quad\;\; 24\,x^4-16\,x^3+16\,x^2 & \\
\hline
6\,x^3-4\,x^2+8\,x-1 \;\big|\; -\;6\,x^3+\;4\,x^2 & \\
\qquad\qquad\quad\;\; -\;6\,x^3+\;4\,x^2-8\,x+1 & \\
\hline
\end{array}$$

It will be observed that *each trial-divisor is equal to the preceding complete divisor with its last term doubled.*

If, in Ex. 2, we had written the expression

$$1-8\,x+20\,x^2-22\,x^3+28\,x^4-12\,x^5+9\,x^6,$$

the square root would have been obtained in the form $1-4\,x+2\,x^2-3\,x^3$, which is the negative of $3\,x^3-2\,x^2+4\,x-1$.

EXERCISE 74

Find the square roots of:

1. $x^4+4\,x^3+6\,x^2+4\,x+1$.
2. $4\,a^4-4\,a^3+17\,a^2-8\,a+16$.
3. $25\,x^4-30\,x^3-x^2+6\,x+1$.
4. $9\,x^4+24\,x^3+28\,x^2+16\,x+4$.
5. $36\,n^6+12\,n^4-60\,n^3+n^2-10\,n+25$.
6. $a^4-8\,a^3b+22\,a^2b^2-24\,ab^3+9\,b^4$.
7. $4\,x^4+12\,x^3y+13\,x^2y^2+6\,xy^3+y^4$.
8. $x^6+12\,x^5+36\,x^4-14\,x^3-84\,x^2+49$.
9. $16\,a^8-40\,a^6x^3+a^4x^6+30\,a^2x^9+9\,x^{12}$.
10. $x^6-2\,x^5-x^4+6\,x^3-3\,x^2-4\,x+4$.
11. $4\,a^6-20\,a^5+41\,a^4-52\,a^3+46\,a^2-24\,a+9$.

175. Square Root of Arithmetical Numbers. — The square root of 100 is 10; of 10000 is 100; etc.

Hence, the square root of a number between 1 and 100 is between 1 and 10; the square root of a number between 100 and 10000 is between 10 and 100; etc.

That is, the integral part of the square root of an integer of one or two digits, contains *one* digit; of an integer of three or four digits, contains *two* digits; and so on.

Hence, if a point be placed over every second digit of an integer, beginning at the units' place, the number of points shows the number of digits in the integral part of its square root.

176. Square Root of any Integral Perfect Square.

The square root of an integral perfect square may be found in the same way as the square root of a polynomial.

Required the square root of 106929.

$$\begin{array}{rrl|l} & & 1\dot{0}6\dot{9}2\dot{9} & 300+20+7 \\ & & a^2=90000 & =a+b+c \\ \hline 2\,a+b= & 600+20 & 16929 & \\ & 20 & 12400 & \\ \hline 2\,a+2\,b+c= & 600+40+7 & 4529 & \\ & 7 & 4529 & \\ \hline \end{array}$$

Pointing the number in accordance with the rule of § 175, we find that there are three digits in its square root.

Let a represent the hundreds' digit of the root, with two ciphers annexed; b the tens' digit, with one cipher annexed; and c the units' digit.

Then, a must be the greatest multiple of 100 whose square is less than 106929; this we find to be 300.

Subtracting a^2, or 90000, from the given number, the result is 16929.

Dividing this remainder by $2\,a$, or 600, we have the quotient 28+; which suggests that b equals 20.

Adding this to $2\,a$, or 600, and multiplying the result by b, or 20, we have 12400; which, subtracted from 16929, leaves 4529.

Since this remainder equals $(2a+2b+c)c$ (§ 172, III), we can get c approximately by dividing it by $2a+2b$, or $600+40$.

Dividing 4529 by 640, we have the quotient $7+$; which suggests that c equals 7.

Adding this to $600+40$, multiplying the result by 7, and subtracting the product, 4529, there is no remainder.

Then, $300+20+7$, or 327, is the required square root.

177. Omitting the ciphers for the sake of brevity, and condensing the operation, we may arrange the work of the example of § 176 as follows:

$$\begin{array}{r|l} 10\dot{6}9\dot{2}\dot{9} & 327 \\ 9 & \\ \hline 62 & 169 \\ & 124 \\ \hline 647 & 4529 \\ & 4529 \end{array}$$

The numbers 600 and 640 are called *trial-divisors*, and the numbers 620 and 647 are called *complete divisors*.

We then have the following rule for finding the square root of an integral perfect square:

Separate the number into periods by pointing every second digit, beginning with the units' place.

Find the greatest square in the left-hand period, and write its square root as the first digit of the root; subtract the square of the first root-digit from the left-hand period, and to the result annex the next period.

Divide this remainder, omitting the last digit, by twice the part of the root already found, and annex the quotient to the root, and also to the trial-divisor.

Multiply the complete divisor by the root-digit last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

Note 1. It sometimes happens that, on multiplying a complete divisor by the digit of the root last obtained, the product is greater than the remainder. In such a case, the digit of the root last obtained is too great, and one less must be substituted for it.

Note 2. If any root-digit is 0, annex 0 to the trial-divisor, and annex to the remainder the next period. (See the illustrative example of § 179.)

178. ***Ex.*** **Find the square root of 4624.**

$$\begin{array}{r|l} \dot{4}6\dot{2}\dot{4} & 68 \\ 36 & \\ \hline 128 \mid 1024 & \\ \quad\quad 1024 & \end{array}$$

The greatest square in the left-hand period is 36.

Then the first digit of the root is 6.

Subtracting 6^2, or 36, from the left-hand period, the result is 10; to this we annex the next period, 24.

Dividing this remainder, omitting the last digit, or 102, by twice the part of the root already found, or 12, the quotient is 8; this we annex to the root, and also to the trial-divisor.

Multiplying the complete divisor, 128, by 8, and subtracting the product from the remainder, there is no remainder.

Then, 68 is the required square root.

179. To find the square root of a number which is not integral.

Ex. **Find the square root of 49.449024.**

We have, $\sqrt{49.449024} = \sqrt{\frac{49449024}{1000000}} = \frac{\sqrt{49449024}}{\sqrt{1000000}}$.

$$\begin{array}{r|l} \dot{4}9\dot{4}4\dot{9}0\dot{2}\dot{4} & 7032 \\ 49 & \\ \hline 1403 \mid 4490 & \\ \quad\quad 4209 & \\ \hline 14062 \mid 28124 & \\ \quad\quad 28124 & \end{array}$$

Since 14 is not contained in 4, we write 0 as the second root-digit, in the above example; we then annex 0 to the trial-divisor 14, and annex to the remainder the next period, 90. (See Note 2, § 177.)

Then, $\sqrt{49.449024} = \frac{7032}{1000} = 7.032.$

The work may be arranged as follows:

$$\begin{array}{r|l} 4\dot{9}.4\dot{4}9\dot{0}2\dot{4} & 7.032 \\ 49 & \\ \hline 1403 \mid 4490 & \\ \quad\quad 4209 & \\ \hline 14062 \mid 28124 & \\ \quad\quad 28124 & \end{array}$$

Then, if a point be placed over every second digit of any number, beginning with the units' place, and extending in either direction, the rule of § 177 may be applied to the result and the decimal point inserted in its proper position in the root.

EXERCISE 75

Find the square roots of the following:

1. 5776.	5. 508369.	9. 3956.41.
2. 15376.	6. 65.1249.	10. 96.4324.
3. 67081.	7. .156816.	11. .00321489.
4. 21904.	8. .064516.	

180. Approximate Square Roots.

If there is a final remainder, the number has no exact square root; but we may continue the operation by annexing periods of ciphers, and obtain an approximate root, correct to any desired number of decimal places.

Ex. Find the square root of 12 to four decimal places.

$$\begin{array}{r|l} 12.00000000 & 3.4641+ \\ 9 & \\ \hline 64 \mid 300 & \\ 256 & \\ \hline 686 \mid 4400 & \\ 4116 & \\ \hline 6924 \mid 28400 & \\ 27696 & \\ \hline 69281 \mid 70400 & \end{array}$$

181. The approximate square root of a fraction may be found by taking the square root of the numerator, and then of the denominator, and dividing the first result by the second.

If the denominator is not a perfect square, it is better to reduce the fraction to an equivalent fraction whose denominator is a perfect square.

Ex. Find the value of $\sqrt{\frac{3}{8}}$ to five decimal places.

We have, $\sqrt{\frac{3}{8}} = \sqrt{\frac{6}{16}} = \frac{\sqrt{6}}{\sqrt{16}} = \frac{2.44948+}{4} = .61237+.$

EXERCISE 76

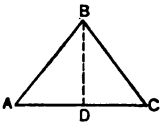
Find the first five figures of the square root of:

*1. 2.	5. .3.	9. 11.	13. 48.
2. 3.	6. $\frac{2}{5}$.	10. 12.	14. 50.
3. 5.	7. $\frac{8}{9}$.	11. .067.	15. $\frac{10}{27}$.
4. 7.	8. $\frac{7}{2}$.	12. $\frac{33}{25}$.	16. .056.

17. .00074.

18. The side of a square is 5; find the diagonal correct to four decimal places.

19. In an equilateral triangle, ABC, the altitude, DB, passes through the middle point of the base. If one side of the triangle is 8, find the altitude, correct to three decimal places.



182. Cube of a Binomial. — We find by actual multiplication:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$\underline{a + b}$$
$$a^3 + 2a^2b + ab^2$$
$$\underline{a^2b + 2ab^2 + b^3}$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

That is, the cube of the sum of two numbers is equal to the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

Again,
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$\underline{a - b}$$
$$a^3 - 2a^2b + ab^2$$
$$\underline{-a^2b + 2ab^2 - b^3}$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

* The values of examples (1) and (2) are of frequent occurrence and are important.

That is, the cube of the difference of two numbers is equal to the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.

1. Find the cube of $a+2b$.

We have, $(a+2b)^3 = a^3+3a^2(2b)+3a(2b)^2+(2b)^3$
$= a^3+6a^2b+12ab^2+8b^3.$

2. Find the cube of $2x^3-5y^2$.

$(2x^3-5y^2)^3 = (2x^3)^3-3(2x^3)^2(5y^2)+3(2x^3)(5y^2)^2-(5y^2)^3$
$= 8x^9-60x^6y^2+150x^3y^4-125y^6.$

The cube of a *trinomial* may be found by the above method, if two of its terms be enclosed in parentheses; and regarded as a single term.

3. Find the cube of x^2-2x-1.

$(x^2-2x-1)^3 = [(x^2-2x)-1]^3$
$= (x^2-2x)^3-3(x^2-2x)^2+3(x^2-2x)-1$
$= x^6-6x^5+12x^4-8x^3-3(x^4-4x^3+4x^2)+3(x^2-2x)-1$
$= x^6-6x^5+12x^4-8x^3-3x^4+12x^3-12x^2+3x^2-6x-1$
$= x^6-6x^5+9x^4+4x^3-9x^2-6x-1.$

EXERCISE 77

Cube each of the following:

1. a^2b-ab^2.
2. $x+4$.
3. $c-5$.
4. $3x^2+1$.
5. $\frac{x}{3}+\frac{3}{x}$.
6. $8x^2-3y^2$.
7. $2m-b$.
8. $9x^2-4y^2$.
9. $a+b+c$.
10. $a-2b-3c$.
11. $3d+4c^2+k$.
12. $3a^x+2b^y$.

183. Cube Root of a Polynomial. The cube roots of certain polynomials of the form

$$a^3+3a^2b+3ab^2+b^3$$

can be found by inspection.

Ex. Find the cube root of $8a^3-36a^2b^2+54ab^4-27b^6$.

We can write the expression as follows:

$$(2a)^3-3(2a)^2(3b^2)+3(2a)(3b^2)^2-(3b^2)^3.$$

By § 182, this is the cube of $2a-3b^2$.

Then, the cube root of the expression is $2a-3b^2$.

EXERCISE 78

Find the cube roots of the following:

1. $x^3+6x^2+12x+8$.
2. $27a^3-27a^2+9a-1$.
3. $m^6+15m^4+75m^2+125$.
4. $a^3-12a^2b+48ab^2-64b^3$.
5. $125x^3+150x^2y+60xy^2+8y^3$.
6. $216a^3-108a^2b+18ab^2-b^3$.
7. $27x^6-135x^5+225x^4-125x^3$.
8. $64t^3-144t^2u+108tu^2-27u^3$.
9. $8h^3+60h^2k+150hk^2+125k^3$.
10. $1-18x^2+108x^4-216x^6$.

XIII. THEORY OF EXPONENTS. IRRATIONAL NUMBERS

184. In the preceding portions of the work, an exponent has been considered only as a *positive integer*.

Thus, if m is a positive integer,

$$a^m = a\times a\times a\times \cdots \text{ to } m \text{ factors.} \qquad (\S\,6)$$

The following results have been proved to hold for any positive integral values of m and n:

$$a^m\times a^n = a^{m+n} \ (\S\,50). \qquad (1)$$

$$(a^m)^n = a^{mn} \ (\S\,85). \qquad (2)$$

185. It is desirable to use exponents which are not positive integers; and we now proceed to assign to them the most convenient definitions and then prove the rules for their use. New meanings are conformed to the old laws. Thus our new exponents are to obey the old index law

$$a^m\times a^n = a^{m+n}. \qquad (1)$$

$$(a^m)^n = a^{mn}. \qquad (2)$$

Let it be required to find such a meaning for *fractional*, *negative* and *zero* exponents.

186. Meaning of a Fractional Exponent.

Let it be required to find a meaning for $a^{\frac{5}{3}}$.

If (1), § 184, is to hold for all values of m and n,

$$a^{\frac{5}{3}} \times a^{\frac{5}{3}} \times a^{\frac{5}{3}} = a^{\frac{5}{3}+\frac{5}{3}+\frac{5}{3}} = a^5.$$

Then, the *third power* of $a^{\frac{5}{3}}$ equals a^5.

Hence, $a^{\frac{5}{3}}$ may be defined as the *cube root* of a^5, or $a^{\frac{5}{3}} = \sqrt[3]{a^5}$.

Consider the general case:

Let it be required to find the meaning of $a^{\frac{p}{q}}$, where p and q are any positive integers.

If (1), § 184, is to hold for all values of m and n,

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots \text{ to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots \text{ to } q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p.$$

Then, the qth power of $a^{\frac{p}{q}}$ equals a^p.

Hence, $a^{\frac{p}{q}}$ must be the qth root of a^p, or $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Hence, in a fractional exponent, the numerator denotes a power, and the denominator a root.

For example, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $b^{\frac{5}{2}} = \sqrt{b^5}$; $x^{\frac{1}{3}} = \sqrt[3]{x}$; etc.

This statement indicates that in expressions affected by a fractional exponent, both a root and a power are to be taken.

EXERCISE 79

Express the following with radical signs:

1. $a^{\frac{5}{6}}$. 3. $7\,m^{\frac{1}{2}}$. 5. $a^{\frac{2}{3}}b^{\frac{2}{5}}$. 7. $8\,a^{\frac{4}{5}}m^{\frac{3}{5}}$. 9. $x^{\frac{2}{4}}y^{\frac{1}{9}}z^{\frac{11}{6}}$.

2. $x^{\frac{7}{2}}$. 4. $5\,x^{\frac{8}{3}}$. 6. $x^{\frac{6}{7}}y^{\frac{10}{9}}$. 8. $10\,n^{\frac{5}{3}}x^{\frac{12}{7}}$. 10. $2\,a^{\frac{1}{m}}b^{\frac{n}{2}}c^{\frac{3p}{q}}$.

Express the following with fractional exponents:

11. $\sqrt[7]{x^6}$. 13. $\sqrt{m^3}$. 15. $3\sqrt[4]{b^9}$. 17. $9\sqrt[5]{m}\sqrt[8]{n^5}$.

12. $\sqrt[6]{a}$. 14. $\sqrt[3]{n^4}$. 16. $4\sqrt[9]{y^2}$. 18. $\sqrt[6]{x^7}\sqrt[9]{y^8}$.

19. $\sqrt[8]{a}\sqrt[7]{b^{10}}$. 20. $\sqrt[3]{x^m}\sqrt{y^{3n}}\sqrt[2p]{z^q}$.

187. Meaning of a Zero Exponent.

If (1), § 184, is to hold for all values of m and n, we have

$$a^m \times a^0 = a^{m+0} = a^m.$$

Whence, $$a^0 = \frac{a^m}{a^m} = 1.$$

This meaning may be illustrated as follows:

Arithmetically, $$\frac{a^3}{a^3} = 1,$$

Algebraically, $$a^3 \div a^3 = a^0.$$

Therefore, $$a^0 = 1.$$ (§ 4, ax. 4)

We must then define a^0 as being equal to 1.

188. Meaning of a Negative Exponent.

Let it be required to find a meaning for a^{-3}.

If (1), § 184, is to hold for all values of m and n,

$$a^{-3} \times a^3 = a^{-3+3} = a^0 = 1 \text{ (§ 187)}.$$

Whence, $$a^{-3} = \frac{1}{a^3}.$$

Consider the general case:

Let it be required to find the meaning of a^{-s}, where s represents a positive integer or a positive fraction.

If (1), § 184, is to hold for all values of m and n,

$$a^{-s} \times a^s = a^{-s+s} = a^0 = 1 \text{ (§ 187)}.$$

Whence, $$a^{-s} = \frac{1}{a^s}.$$

We must then define a^{-s} as being equal to 1 divided by a^s.

For example, $a^{-2} = \frac{1}{a^2}$; $a^{-\frac{3}{2}} = \frac{1}{a^{\frac{3}{2}}}$; $3x^{-1}y^{-\frac{1}{2}} = \frac{3}{xy^{\frac{1}{2}}}$; etc.

189. It follows from § 188 that

Any *factor* of the numerator of a fraction may be transferred to the denominator, or any *factor* of the denominator to the numerator, if the sign of its exponent be changed.

Thus, $\frac{a^2b^3}{cd^4} = \frac{b^3}{a^{-2}cd^4} = \frac{a^2b^3c^{-1}}{d^4} = \frac{a^2d^{-4}}{b^{-3}c}$, etc.

EXERCISE 80

Express with positive exponents:

1. $x^{-4}y^3$.
2. $a^{\frac{5}{7}}b^{-8}$.
3. $m^{-\frac{1}{4}}n^{\frac{1}{6}}$.
4. $3\,n^{-1}x$.
5. $a^{-2}m^{-3}$.
6. $m^{-1\frac{3}{5}}x^9$.
7. $4\,a^{-\frac{4}{3}}n^{-5}$.
8. $5\,x^4y^{-\frac{8}{9}}z^{-7}$.
9. $m^{-\frac{2}{7}}n^{-9}$.
10. $8\,a^{-\frac{5}{4}}b^{-10}c^7$.
11. $6\,m^{-\frac{2}{3}}n^{-\frac{7}{6}}x^{\frac{3}{4}}$.
12. $7\,a^{-\frac{5}{2}}n^{-6}x^{-\frac{6}{5}}$.

Transfer all *literal* factors from the denominators to the numerators in the following:

13. $\frac{1}{x^8}$.
14. $\frac{a^{\frac{4}{5}}}{b^{-\frac{7}{3}}}$.
15. $\frac{1}{8\,m^{-2}n^{\frac{3}{2}}}$.
16. $\frac{3}{ax^{-4}}$.
17. $\frac{z^{-\frac{2}{5}}}{x^{-\frac{2}{7}}y^3}$.
18. $\frac{2\,m^5}{5\,np^{-1}}$.
19. $\frac{7\,a^{\frac{1}{3}}b^{-3}}{6\,c^{\frac{1}{6}}d^{\frac{8}{9}}}$.
20. $\frac{9m^{-7}x^{-\frac{5}{4}}}{4\,n^{-\frac{5}{8}}y^{-9}}$.

Transfer all literal factors from the numerators to the denominators in the following:

21. $\frac{4\,a^7}{3}$.
22. $\frac{x^{\frac{4}{5}}}{y^{\frac{3}{7}}}$.
23. $\frac{m^{-2}n^{-\frac{1}{6}}}{2}$.
24. $\frac{xy^{-5}}{5}$.
25. $\frac{6\,a^{-\frac{2}{3}}b^3}{c^6}$.
26. $\frac{a^8m^{\frac{3}{7}}}{n^{-7}}$.
27. $\frac{9\,m^{-6}n^{-\frac{7}{8}}}{x^9y^{-\frac{1}{3}}}$.
28. $\frac{8\,a^{-1}b^{\frac{8}{9}}}{5\,c^{-8}d^{-\frac{7}{8}}}$.

190. Since this is an elementary course, the student is only expected to *read* §§ 191 to 194, then use § 196 in applying the principles involved.

191. We obtained the definitions of fractional, zero, and negative exponents by supposing equation (1), § 184, to hold for such exponents.

Then, for any values of m and n,

$$a^m \times a^n = a^{m+n}. \qquad (1)$$

The formal proof of this result for positive or negative, integral or fractional, values of m and n will be found in the Second Course.

192. $\frac{a^m}{a^n} = a^{m-n}$ *for all values of* m *and* n.

By § 189, $\frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}$, by (1), § 191.

The proof of this result in the case where m and n are positive integers, and $m > n$, is given in § 63.

193. To prove equation (2), § 184, for any values of m and n, considering three cases, in each of which m may have any value, positive or negative, integral or fractional.

I. Let n be a positive integer.

The proof given in § 85 holds if n is a positive integer, whatever the value of m.

II. Let $n = \frac{p}{q}$, where p and q are positive integers.

Then, by the definition of § 186,

$$(a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} \text{ (§ 193, I)} = a^{\frac{mp}{q}}.$$

III. Let $n = -s$, where s is a positive number.

Then, by the definition of § 188,

$$(a^m)^{-s} = \frac{1}{(a^m)^s} = \frac{1}{a^{ms}} \text{ (§ 193, I or II)} = a^{-ms}.$$

Therefore, the result holds for all values of m and n.

194. To prove the result

$$(ab)^n = a^n b^n,$$

for any fractional or negative value of n.

The proof of this result in the case where n is any positive integer, was given in § 86.

I. Let $n = \frac{p}{q}$, where p and q are any positive integers.

By § 193, $$[(ab)^{\frac{p}{q}}]^q = (ab)^p = a^p b^p \text{ (§ 86)}. \qquad (1)$$

By § 86, $$(a^{\frac{p}{q}} b^{\frac{p}{q}})^q = (a^{\frac{p}{q}})^q (b^{\frac{p}{q}})^q = a^p b^p. \qquad (2)$$

From (1) and (2), $[(ab)^{\frac{p}{q}}]^q = (a^{\frac{p}{q}} b^{\frac{p}{q}})^q$.

Taking the qth root of both members, we have

$$(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}.$$

II. Let $n = -s$, where s is any positive integer or positive fraction.

Then, $(ab)^{-s} = \frac{1}{(ab)^s} = \frac{1}{a^s b^s}$ (§ 86, or § 194, I) $= a^{-s} b^{-s}$.

195. The value of a numerical expression affected with a fractional exponent may be found by first, if possible, extracting the root indicated by the denominator, and then raising the result to the power indicated by the numerator.

Ex. Find the value of $(-8)^{\frac{2}{3}}$.

By § 193, $(-8)^{\frac{2}{3}} = [(-8)^{\frac{1}{3}}]^2 = (\sqrt[3]{-8})^2 = (-2)^2 = 4$.

This holds only for *real* numbers.

196. Remember that the exponent laws given in §§ 50, 63, 88, 168, hold whether the exponents be integral or fractional either positive or negative, i.e. In multiplication, add exponents of like letters; in division, subtract exponents of like letters in the divisor from those in the dividend; in involution, multiply the exponents by the index of the power; in evolution, divide the exponents by the index of the root.

1. Find the value of $a^2 \times a^{-5}$.

We have, $a^2 \times a^{-5} = a^{2-5} = a^{-3}$.

2. Find the value of $a \times \sqrt{a^5}$.

By § 186, $a \times \sqrt{a^5} = a \times a^{\frac{5}{2}} = a^{1+\frac{5}{2}} = a^{\frac{7}{2}}$.

3. Multiply $a + 2\,a^{\frac{2}{3}} - 3\,a^{\frac{1}{3}}$ by $2 - 4\,a^{-\frac{1}{3}} - 6\,a^{-\frac{2}{3}}$.

$$\begin{array}{l}
a + 2\,a^{\frac{2}{3}} - 3\,a^{\frac{1}{3}} \\
\underline{2 - 4\,a^{-\frac{1}{3}} - 6\,a^{-\frac{2}{3}}} \\
2\,a + 4\,a^{\frac{2}{3}} - 6\,a^{\frac{1}{3}} \\
\quad -4\,a^{\frac{2}{3}} - 8\,a^{\frac{1}{3}} + 12 \\
\underline{\qquad\quad - 6\,a^{\frac{1}{3}} - 12 + 18\,a^{-\frac{1}{3}}} \\
2\,a \qquad -20\,a^{\frac{1}{3}} \qquad +18\,a^{-\frac{1}{3}}
\end{array}$$

It must be carefully observed, in examples like the foregoing, that the zero power of any number equals 1 (§ 187).

4. Find the value of $\dfrac{a^{-3}}{\sqrt[5]{a^2}}$.

$$\frac{a^{-3}}{\sqrt[5]{a^2}}=\frac{a^{-3}}{a^{\frac{2}{5}}}=a^{-3-\frac{2}{5}}=a^{-\frac{17}{5}}.$$

5. Divide $18\,xy^{-2}-23+x^{-\frac{1}{2}}y+6\,x^{-1}y^2$ by $3\,x^{\frac{1}{2}}y^{-1}+x^{\frac{1}{4}}-2\,x^{-\frac{1}{4}}y$.

$$\begin{array}{l|l}
18\,xy^{-2}-23+x^{-\frac{1}{2}}y+6\,x^{-1}y^2 & 3\,x^{\frac{1}{2}}y^{-1}+x^{\frac{1}{4}}-2\,x^{-\frac{1}{4}}y \\
\underline{18\,xy^{-2}+6\,x^{\frac{1}{2}}y^{-1}-12} & 6\,x^{\frac{1}{2}}y^{-1}-2\,x^{-\frac{1}{4}}-3\,x^{-\frac{3}{4}}y \\
\quad -6\,x^{\frac{1}{2}}y^{-1}-11+\ x^{-\frac{1}{2}}y+6\,x^{-1}y^2 & \\
\quad \underline{-6\,x^{\frac{1}{2}}y^{-1}-\ 2+4\,x^{-\frac{1}{2}}y} & \\
\qquad -\ 9-3\,x^{-\frac{1}{2}}y+6\,x^{-1}y^2 & \\
\qquad \underline{-\ 9-3\,x^{-\frac{1}{2}}y+6\,x^{-1}y^2} &
\end{array}$$

It is important to arrange the dividend, divisor, and each remainder in the same order of powers of some common letter.

6. Find the value of $(a^2)^{-5}$.

We have, § 193, III, $(a^2)^{-5}=a^{2\times-5}=a^{-10}$.

7. Find the value of $(a^{-3})^{-\frac{1}{3}}$.

$$(a^{-3})^{-\frac{1}{3}}=a^{-3\times-\frac{1}{3}}=a.$$

8. Find the value of $(\sqrt{a})^{\frac{2}{3}}$.

$$(\sqrt{a})^{\frac{2}{3}}=(a^{\frac{1}{2}})^{\frac{2}{3}}=a^{\frac{1}{2}\times\frac{2}{3}}=a^{\frac{1}{3}}.$$

EXERCISE 81

Multiply the following:

1. a^4 by $a^{\frac{1}{2}}$.
2. a^6 by a^{-4}.
3. a^{-5} by a^3.
4. $2\,a^5$ by a^{-5}.
5. $4\,a^{\frac{1}{3}}$ by $6\,a^{\frac{2}{3}}$.
6. $12\,x^{-1}$ by $\sqrt{x}$.
7. $a^{-5}\sqrt[4]{x^3}$ by $a^{-3}\sqrt[3]{x^4}$.
8. $m^{-1}k^{-\frac{3}{10}}$ by $\dfrac{1}{3\,m^{-\frac{2}{7}}}k^{\frac{1}{5}}$.
9. $\sqrt[6]{a}$ by $a^{\frac{3}{4}}$.
10. $3\,x^{-\frac{1}{2}}y$ by $4\,x^{\frac{1}{2}}y^{-1}$.

Divide the following:

11. x^4 by x^6.
12. $a^{\frac{5}{3}}$ by $a^{\frac{1}{2}}$.
13. c by $c^{-\frac{4}{7}}$.
14. $4x^{-4}$ by $7x^{-6}$.
15. $6\sqrt{x^3}$ by $3\sqrt{x}$.
16. $3x^{-\frac{2}{3}}y^{-1}$ by $4x^{-\frac{1}{2}}y$.
17. $12a^{-2}b^4c$ by $6a^{-2}d$.
18. $14a^{-1}b^{\frac{1}{3}}$ by $-7ab^{\frac{1}{3}}$.

Simplify:

19. $(a^{\frac{1}{2}}+a^{\frac{1}{4}}+1)a^{-\frac{1}{2}}$.
20. $(x^{-2}+2x^{-1}+1)4x^2$.
21. $(4a^{-4}+10a^{-2}+25)(2a^{-2}-5)$.
22. $\dfrac{x^{-4}+2a^{-1}x^{-2}-15a^{-2}}{x^{-2}+5a^{-1}}$.
23. $(x^2-1)\div(x^{\frac{1}{2}}+1)$.
24. $n^{-6}-n^{-3}\div n^{-3}-1$.
25. $(n^{-6}-n^{-3})\div(n^{-3}-1)$.
26. $(2n^{-\frac{3}{4}}-5-6n^{\frac{3}{4}})(3n^{-\frac{3}{4}}-4)$.
27. $(a^{-\frac{4}{3}}+2a^{-\frac{2}{3}}b^{-2}+9b^{-4})\div(a^{-1}+2a^{-\frac{2}{3}}b^{-1}+3a^{-\frac{1}{3}}b^{-2})$.

Find by inspection:

28. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}})$. (§ 89.)
29. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2$. (§ 91.)
30. $(a^{-1}+2b^{-2})(a^{-1}-6b^{-2})$.
31. $(2a^{\frac{1}{3}}+3b^{\frac{1}{4}})(5a^{\frac{1}{3}}+3b^{\frac{1}{4}})$.
32. $(a+8b)\div(a^{\frac{1}{3}}+2b^{\frac{1}{3}})$.
33. $(16a-25c)\div(4a^{\frac{1}{2}}-5c^{\frac{1}{2}})$.

34. Factor $4a-4a^{\frac{1}{2}}+1$. (Call the first term a perfect square.)

35. $49-36b$. Factor, using § 89.
36. $(c^{\frac{1}{3}}-2d^{\frac{1}{3}})^3=?$

Supply the missing term in the following trinomial squares:

37. $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}$.
38. $9c+12c^{\frac{1}{2}}$.
39. $x+9$.
40. $25a^{\frac{1}{2}}-10a^{\frac{1}{4}}$.
41. $16c^{\frac{1}{2}}+36d^{\frac{1}{2}}$.

Find the values of:

42. $(x^{-\frac{2}{3}}y^{\frac{6}{5}})^{\frac{1}{2}}$.
43. $(a^{\frac{1}{4}}x^{\frac{3}{2}})^{-\frac{4}{3}}$.
44. $(n^{-3}\sqrt{x^6})^{-4}$.
45. $(\sqrt[4]{a^3}\sqrt[5]{b^{-8}})^{\frac{5}{6}}$.
46. $81^{-\frac{1}{2}}$.
47. $(-32)^{\frac{4}{5}}$.
48. $36^{-\frac{3}{2}}$.
49. $(-8)^{\frac{10}{3}}$.
50. $729^{-\frac{5}{6}}$.
51. $512^{\frac{5}{9}}$.

Extract the square root of:

52. $16\,a^{-6}m^{\frac{1}{2}}$.

53. $4\,a^{-\frac{8}{5}}+20\,a^{-\frac{6}{5}}+21\,a^{-\frac{4}{5}}-10\,a^{-\frac{2}{5}}+1$.

54. $4\,a^{\frac{1}{2}}+4\,a^{\frac{1}{4}}-19-10\,a^{-\frac{1}{4}}+25\,a^{-\frac{1}{2}}$.

Simplify the following, expressing all the results with positive exponents:

55. $\dfrac{a^{-4}b^2}{c^{-1}}\cdot\dfrac{c^2}{ab}$.

56. $\dfrac{a^n\cdot(a^{n-1})^n}{a^{n+1}\cdot a^{n-1}}$.

57. $\dfrac{a^{\frac{9}{10}}}{b^{\frac{5}{6}}c^{\frac{1}{3}}}\cdot\dfrac{b^{\frac{3}{4}}c^{\frac{7}{12}}}{a^{\frac{3}{15}}}$.

58. $(2^{n+4}-2\cdot 2^n)(2^{-2}\cdot 2^{-n-2})$.

59. $\dfrac{a+b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-\dfrac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a-b}$.

60. $\dfrac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}-\dfrac{a^{-\frac{1}{2}}+b^{-\frac{1}{2}}}{a^{-\frac{1}{2}}-b^{-\frac{1}{2}}}$.

61. $(a^{\frac{1}{2}}+a^{-\frac{1}{2}})^2-4=\,?$

62. $\dfrac{\left(\dfrac{e^x+e^{-x}}{2}\right)^2-1}{\left(\dfrac{e^x-e^{-x}}{2}\right)^2}=\,?$

Extract the square root of:

63. $a+2\,a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.

64. $2+2\sqrt{6}+3$. (See § 91.)

65. Is $6+2(18)^{\frac{1}{2}}+3$ a perfect square?

IRRATIONAL NUMBERS

197. A **Surd** is the indicated root of a number, or expression, which is not a perfect power of the degree denoted by the index of the radical sign; as $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{x+y}$, or $(3)^{\frac{1}{2}}$.

198. A monomial is said to be *rational* when it is rational and integral (§ 57), or else a fraction whose terms are rational and integral.

A polynomial is said to be rational when each of its terms is rational.

An expression is said to be *irrational* when it involves surds; as $2+\sqrt{3}$, or $\sqrt{a+1}-\sqrt{a}$.

199. A *rational number* is a positive or negative integer, or a positive or negative fraction.

A numerical expression involving surds is an *irrational number.**

200. If a surd is in the form $b\sqrt[n]{a}$, b is called the *coefficient* of the surd, and n the *index*.

201. The *degree* of a surd is denoted by its index; thus, $\sqrt[3]{5}$ is a surd of the third degree.

A *quadratic surd* is a surd of the second degree.

For example, the square roots of positive arithmetical numbers *not* belonging to the set

1, 4, 9, 16, 25, 36, 49, etc., are *quadratic surds*.

The cube roots of arithmetical numbers *not* belonging to the set

1, 8, 27, 64, 125, etc., are *cubic surds*.

REDUCTION OF A SURD TO ITS SIMPLEST FORM

202. A surd is said to be in its *simplest form* when the expression under the radical sign is rational and integral (§ 57), is not a perfect power of the degree denoted by any factor of the index of the surd, and has no factor which is a perfect power of the same degree as the surd.

203. CASE I. *When the expression under the radical sign is a perfect power of the degree denoted by a factor of the index.*

Ex. 1. Reduce $\sqrt[6]{8}$ to its simplest form.

We have, $\sqrt[6]{8} = \sqrt[6]{2^3} = 2^{\frac{3}{6}}$ (§ 186) $= 2^{\frac{1}{2}} = \sqrt{2}$.

Ex. 2. Reduce $\sqrt[6]{16}$ to its simplest form.

We have, $\sqrt[6]{16} = [(2^4)^{\frac{1}{2}}]^{\frac{1}{3}} = (2^2)^{\frac{1}{3}} = 4^{\frac{1}{3}} = \sqrt[3]{4}$.

In many radical forms, operations are more simple when the quantities are reduced to forms with fractional exponents.

* Note that we do not define irrational number. The two most important irrationals, $-\pi$ and e (the base of a system of logarithms), — have been proved not to involve surds.

EXERCISE 82

Reduce the following to their simplest forms:

1. $\sqrt[4]{25}$.
2. $\sqrt[10]{4}$.
3. $\sqrt[8]{121}$.
4. $\sqrt[9]{125}$.
5. $\sqrt[16]{49}$.
6. $\sqrt[16]{81}$.
7. $\sqrt[14]{64}$.
8. $\sqrt[10]{81}$.
9. $\sqrt[10]{243}$.
10. $\sqrt[15]{343}$.
11. $\sqrt[4]{144\,x^6y^8}$.
12. $\sqrt[6]{27\,n^6x^{12}}$.
13. $\sqrt[12]{216\,a^9x^3}$.
14. $\sqrt[12]{64\,a^6b^{18}}$.
15. $\sqrt[18]{8\,a^{18}b^{15}}$.
16. $\sqrt[12]{625\,x^{12}y^8}$.

204. CASE II. *When the expression under the radical sign is rational and integral, and has a factor which is a perfect power of the same degree as the surd.*

1. Reduce $\sqrt[3]{54}$ to its simplest form.

We have, $\sqrt[3]{54}=(27\cdot 2)^{\frac{1}{3}}=27^{\frac{1}{3}}\cdot 2^{\frac{1}{3}}=(3^3)^{\frac{1}{3}}\cdot 2^{\frac{1}{3}}=3\cdot 2^{\frac{1}{3}}=3\sqrt[3]{2}$.

2. Reduce $\sqrt{3\,a^3b-12\,a^2b^2+12\,ab^3}$ to its simplest form.

$$\sqrt{3\,a^3b-12\,a^2b^2+12\,ab^3}=\sqrt{(a^2-4\,ab+4\,b^2)3\,ab}$$
$$=\sqrt{a^2-4\,ab+4\,b^2}\sqrt{3\,ab}=(a-2\,b)\sqrt{3\,ab}.$$

We then have the following rule:

Resolve the expression under the radical sign into two factors, the second of which contains no factor which is a perfect power of the same degree as the surd.

Extract the required root of the first factor, and multiply the result by the indicated root of the second.

If the expression under the radical sign has a numerical factor which cannot be readily factored by inspection, it is convenient to resolve it into its prime factors.

3. Reduce $\sqrt[3]{1944}$ to its simplest form.

$$\sqrt[3]{1944}=\sqrt[3]{2^3\times 3^5}=(2^3\cdot 3^5)^{\frac{1}{3}}=(2^3\cdot 3^3)^{\frac{1}{3}}\cdot(3^2)^{\frac{1}{3}}=2\cdot 3\cdot(3^2)^{\frac{1}{3}}=6\sqrt[3]{9}.$$

4. Reduce $\sqrt{125\times 147}$ to its simplest form.

$$\sqrt{125\times 147}=\sqrt{5^3\times 3\times 7^2}=\sqrt{5^2\times 7^2}\times\sqrt{5\times 3}=5\times 7\times\sqrt{15}=35\sqrt{15}.$$

EXERCISE 83

Reduce the following to their simplest forms:

1. $(45)^{\frac{1}{2}}$. 4. $\sqrt{75}$. 7. $\sqrt[3]{54}$. 10. $\sqrt[5]{64}$.
2. $(12)^{\frac{1}{2}}$. 5. $\sqrt{15}$. 8. $(128)^{\frac{1}{3}}$. 11. $\sqrt{500\,a^3b^6}$.
3. $(96)^{\frac{1}{2}}$. 6. $(27)^{\frac{1}{3}}$. 9. $\sqrt[6]{192}$. 12. $\sqrt{98\,x^6y-49\,x^4y^2}$.
13. $[(a+3)\,(a^2-9)]^{\frac{1}{2}}$.
14. $\sqrt{(2\,a^2+2\,ay-4\,y^2)\,(3\,a-3\,y)}$.
15. $[(x^2+9)x]^{\frac{1}{2}}$. 19. $\sqrt[3]{63\,x^2y\cdot 75\,x^2y^3\cdot 98\,z}$.
16. $\sqrt{(x^2+6\,x+9)x}$. 20. $\sqrt{98\cdot 196}$.
17. $\sqrt{(a^2+2\,ab+4\,b^2)a^2}$. 21. $(5145)^{\frac{1}{2}}$.
18. $\sqrt{a^3+4\,ab+4\,b^2(a-b)}$. 22. $\sqrt{3\,a^3-24\,a^2+48\,a}$.
23. $\sqrt{18\,a^3b+60\,a^2b^2+50\,ab^3}$.
24. $\sqrt{(6\,x^2+5\,xy-4\,y^2)\,(3\,x^2-2\,xy-8\,y^2)}$.

205. CASE III. *When the expression under the radical sign is a fraction.*

In this case, we multiply both terms of the fraction by such an expression as will make the denominator a perfect power of the same degree as the surd, and then proceed as in § 204.

Ex. Reduce $\sqrt{\frac{9}{8\,a^3}}$ to its simplest form.

Multiplying both terms of the fraction by $2\,a$, we have

$$\sqrt{\frac{9}{8\,a^3}}=\sqrt{\frac{9\times 2\,a}{16\,a^4}}=\sqrt{\frac{9}{16\,a^4}\times 2\,a}=\sqrt{\frac{9}{16\,a^4}}\times\sqrt{2\,a}=\frac{3}{4\,a^2}\sqrt{2\,a}.$$

EXERCISE 84

Reduce the following to their simplest forms:

1. $\sqrt{\frac{5}{2}}$. 3. $\sqrt{2\frac{5}{8}}$. 5. $\sqrt{\frac{23}{27}}$. 7. $\sqrt[3]{\frac{4}{9}}$. 9. $\sqrt[5]{\frac{2}{3}}$.
2. $\sqrt{\frac{2}{5}}$. 4. $\sqrt{\frac{17}{18}}$. 6. $\sqrt{\frac{11}{12}}$. 8. $\sqrt[4]{\frac{8}{8}}$. 10. $\sqrt{\frac{5}{32}}$.

11. $\sqrt{\frac{5}{16}}$. 12. $\sqrt{\frac{2a-b}{2a+b}}$. 13. $\sqrt{\frac{a^2-ab+b^2}{a+b}}$.

14. $\frac{1}{a-3}\sqrt{\frac{2a^2-a-15}{a-3}}$. 15. $\frac{x^3}{x^2-5x+6}\sqrt{\frac{3x^2-18x+27}{x^3}}$.

206. To Introduce the Coefficient of a Surd under the Radical Sign.

The coefficient of a surd may be introduced under the radical sign by raising it to the power denoted by the index.

Ex. Introduce the coefficient of $2\sqrt[3]{3}$ under the radical sign.

$$2\sqrt[3]{3}=\sqrt[3]{8}\times\sqrt[3]{3}=\sqrt[3]{8\times3}=\sqrt[3]{24}.$$

A rational expression (§ 198) may be expressed in the form of a surd of any degree by raising it to the power denoted by the index, and writing the result under the corresponding radical sign.

EXERCISE 85

Introduce the coefficients of the following under the radical signs:

1. $2\sqrt{3}$. 3. $6\sqrt{6}$. 5. $2\sqrt[3]{5}$.
2. $5\sqrt{2}$. 4. $5\sqrt[3]{3}$. 6. $3\sqrt{\frac{1}{3}}$.
7. $(2a+b)\sqrt{\frac{1}{4a^2-b^2}}$.
8. $(3x-2y)\sqrt{x}$.
9. $(a+b)\sqrt{\frac{a-b}{a+b}}$.
10. $\frac{a-3}{a+3}\sqrt{\frac{a^2+a-6}{a^2-a-6}}$.
11. $\frac{1}{2a-3b}\sqrt{4a^2-9b^2}$.

ADDITION AND SUBTRACTION OF SURDS

207. Similar Surds are surds which do not differ at all, or differ only in their coefficients; as $2\sqrt[3]{ax^2}$ and $3\sqrt[3]{ax^2}$.

Dissimilar Surds are surds which are not similar.

208. To add or subtract *similar surds* (§ 207), add or subtract their coefficients, and multiply the result by their common surd part.

1. Required the sum of $\sqrt{20}$ and $\sqrt{45}$.

Reducing each surd to its simplest form (§ 204),

$$\sqrt{20}+\sqrt{45}=\sqrt{4\times5}+\sqrt{9\times5}=2\sqrt{5}+3\sqrt{5}=5\sqrt{5}.$$

2. Simplify $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}$.

$$\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}=\sqrt{\frac{1}{4}\times2}+\sqrt{\frac{1}{9}\times6}-\sqrt{\frac{9}{16}\times2}$$

$$=\frac{1}{2}\sqrt{2}+\frac{1}{3}\sqrt{6}-\frac{3}{4}\sqrt{2}=\frac{1}{3}\sqrt{6}-\frac{1}{4}\sqrt{2}.$$

We then have the following rule:

Reduce each surd to its simplest form.

Add or subtract the similar surds, and indicate the addition or subtraction of the dissimilar.

EXERCISE 86

Simplify the following:

1. $\sqrt{12}+\sqrt{48}$.
2. $\sqrt{50}-\sqrt{18}$.
3. $2\sqrt{2}+\sqrt{128}-\sqrt{98}$.
4. $\sqrt{125}+\sqrt{180}+\sqrt{80}$.
5. $\sqrt[3]{2}+\sqrt[3]{16}$.
6. $\sqrt[3]{40}+\sqrt[3]{135}$.
7. $\sqrt[4]{32}-\sqrt[4]{162}$.
8. $\sqrt{250}+\sqrt{490}-\sqrt{10}$.
9. $\sqrt{44}-\sqrt{275}+\sqrt{891}$.
10. $\sqrt{32}+\sqrt{48}+\sqrt{80}$.
11. $\sqrt{5}+\sqrt{245}-\sqrt{320}$.
12. $\sqrt{\frac{8}{5}}+\sqrt{\frac{9}{10}}-\sqrt{\frac{5}{8}}$.
13. $\sqrt{\frac{16}{15}}-\sqrt{\frac{20}{3}}+\sqrt{\frac{3}{5}}$.
14. $\sqrt[3]{\frac{1}{4}}+\sqrt[3]{\frac{1}{32}}+\sqrt[3]{\frac{2}{3}}$.
15. $\frac{\sqrt{175}}{5}-\frac{\sqrt{112}}{3}-\frac{\sqrt{44}}{2}$.
16. $7\sqrt{27}-\sqrt{75}-24\sqrt{\frac{1}{12}}-27\sqrt{\frac{1}{27}}$.
17. $\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{1}{36}}$.
18. $b^2\sqrt{8\,a^5b}+ab\sqrt{50\,a^3b^3}-a^2\sqrt{128\,ab^5}$.
19. $\sqrt{3\,x^3+12\,x^2+12\,x}+\sqrt{27\,x^3-72\,x^2+48\,x}$.
20. $2\sqrt{12\,x^2+60\,xy+75\,y^2}-\sqrt{48\,x^2-72\,xy+27\,y^2}$.

TO REDUCE SURDS OF DIFFERENT DEGREES TO EQUIVALENT SURDS OF THE SAME DEGREE

209. *Ex.* Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to equivalent surds of the same degree.

By § 186,
$$\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{6}{12}}=\sqrt[12]{2^6}=\sqrt[12]{64}.$$
$$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{4}{12}}=\sqrt[12]{3^4}=\sqrt[12]{81}.$$
$$\sqrt[4]{5}=5^{\frac{1}{4}}=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}.$$

Rule:

Express the surds with fractional exponents, reduce these to their lowest common denominator, and express the resulting expressions with radical signs.

The relative magnitudes of surds may be determined by reducing them, if necessary, to equivalent surds of the same degree.

Thus, in the above example, $\sqrt[12]{125}$ is greater than $\sqrt[12]{81}$, and $\sqrt[12]{81}$ than $\sqrt[12]{64}$.

Then, $\sqrt[4]{5}$ is greater than $\sqrt[3]{3}$, and $\sqrt[3]{3}$ than $\sqrt{2}$.

EXERCISE 87

Reduce the following to equivalent surds of the same degree:

1. $\sqrt{2}$ and $\sqrt[3]{3}$.
2. $\sqrt[3]{5}$ and $\sqrt[4]{7}$.
3. $\sqrt[3]{2}$ and $\sqrt[5]{3}$.
4. $\sqrt[4]{a}$, $\sqrt[7]{b}$, $\sqrt{c}$.
5. $\sqrt{a-b}$, $\sqrt[3]{a+b}$, $\sqrt[4]{a^2-b^2}$.
6. Is $\sqrt{2}$ greater than $\sqrt[3]{3}$?
7. Compare $\sqrt[5]{5}$ and $\sqrt[7]{7}$.
8. Write in order of magnitude $\sqrt[3]{4}$, $\sqrt[4]{6}$, $\sqrt[6]{15}$.
9. Which is greatest $\sqrt{3}$, $\sqrt[5]{15}$, $\sqrt[10]{253}$?
10. Which is greatest $\sqrt[3]{3}$, $\sqrt{2}$, $\sqrt[6]{4}$?

MULTIPLICATION AND EVOLUTION OF SURDS

210. 1. Multiply $\sqrt{6}$ by $\sqrt{15}$.

$$\sqrt{6}\times\sqrt{15}=\sqrt{6\times15}=\sqrt{2\times3\times3\times5}=\sqrt{3^2\times2\times5}=3\sqrt{10}.$$

2. Multiply $\sqrt{2\,a}$ by $\sqrt[3]{4\,a^2}$.

Reducing to equivalent surds of the same degree (§ 209),

$$\sqrt{2a}\times\sqrt[3]{4a^2}=(2a)^{\frac{1}{2}}\times(4a^2)^{\frac{1}{3}}=(2a)^{\frac{3}{6}}\times(4a^2)^{\frac{2}{6}}=\sqrt[6]{(2a)^3}\times\sqrt[6]{(4a^2)^2}$$
$$=\sqrt[6]{2^3a^3\times2^4a^4}=\sqrt[6]{2^6a^6\times2a}=2a\sqrt[6]{2a}.$$

We then have the following rule:

To multiply together two or more surds, reduce them, if necessary, to surds of the same degree.

Multiply together the expressions under the radical signs, and write the result under the common radical sign. ***The result should be reduced to its simplest form.***

3. **Multiply $\sqrt{5}$ by $\sqrt[6]{5}$.**

By § 186, $\sqrt{5}=5^{\frac{1}{2}}=5^{\frac{3}{6}}=\sqrt[6]{5^3}$.

Then, $\sqrt{5}\times\sqrt[6]{5}=\sqrt[6]{5^3}\times\sqrt[6]{5}=\sqrt[6]{5^4}=5^{\frac{4}{6}}=5^{\frac{2}{3}}=\sqrt[3]{5^2}=\sqrt[3]{25}$.

4. **Multiply $2\sqrt{3}+3\sqrt{2}$ by $3\sqrt{3}-\sqrt{2}$.**

$$\begin{array}{r} 2\sqrt{3}+3\sqrt{2} \\ 3\sqrt{3}-\sqrt{2} \\ \hline 18+9\sqrt{6} \\ -2\sqrt{6}-6 \\ \hline 18+7\sqrt{6}-6=12+7\sqrt{6}. \end{array}$$

To multiply a surd of the second degree by itself simply removes the radical sign; thus, $\sqrt{3}\times\sqrt{3}=3$, or $3^{\frac{1}{2}}\cdot3^{\frac{1}{2}}=3^{\frac{1}{2}+\frac{1}{2}}=3$.

5. **Multiply $3\sqrt{1+x}-4\sqrt{x}$ by $\sqrt{1+x}+2\sqrt{x}$.**

$$\begin{array}{r} 3\sqrt{1+x}-4\sqrt{x} \\ \sqrt{1+x}+2\sqrt{x} \\ \hline 3\,(1+x)-4\sqrt{x+x^2} \\ +6\sqrt{x+x^2}-8x \\ \hline 3\,(1+x)+2\sqrt{x+x^2}-8x=3-5x+2\sqrt{x+x^2}. \end{array}$$

EXERCISE 88

Multiply the following:

1. $\sqrt{3}$ by $\sqrt{27}$.
2. $\sqrt[3]{36x^2y}$ by $\sqrt[3]{6xy^2}$.
3. $10^{\frac{1}{2}}$ by $30^{\frac{1}{2}}$.
4. $(108)^{\frac{1}{2}}$ by $(192)^{\frac{1}{2}}$.
5. $\sqrt[3]{72}$ by $\sqrt[3]{12}$.
6. $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{1}{8}}$.

7. $\sqrt{\frac{4}{14}}$ by $\sqrt{\frac{1}{8}}$.

8. $\sqrt{a^2-b^2}$ by $\sqrt{a+b}$.

9. $\sqrt[3]{3}$ by $\sqrt{2}$.

10. $\sqrt[4]{7}$ by $\sqrt{7}$.

11. $\sqrt[3]{7\,ab^2}$ by $\sqrt{7\,ab}$.

12. $\sqrt[3]{5\,a^2}$ by $\sqrt{125\,ab}$.

13. $2+3^{\frac{1}{2}}$ by $3+2^{\frac{1}{2}}$.

14. $5+\sqrt{7}\cdot 2+\sqrt{5}$.

15. $(5+\sqrt{7})(2+\sqrt{5})$.

16. Expand $(\sqrt{a}+\sqrt{b})^2$ (§ 91).

17. Expand $[(a)^{\frac{1}{2}}+7(b)^{\frac{1}{2}}][(a)^{\frac{1}{2}}-5(b)^{\frac{1}{2}}]$.

18. Expand $(\sqrt{3\,a}+\sqrt{4\,b})(\sqrt{3\,a}-\sqrt{4\,b})$.

19. Expand $[3\cdot 2^{\frac{1}{2}}-2\cdot 3^{\frac{1}{2}}]^2$.

20. Expand $(\sqrt{1}+\sqrt{2}+\sqrt{3})^2$.

21. Expand $(2\sqrt{3})^3$, $(3\sqrt[3]{2})^3$, $(4\sqrt[4]{2})^4$.

22. Expand $(\sqrt[6]{12})^3$.

$(\sqrt[6]{12})^3=(12)^{\frac{3}{6}}$ (§ 193) $=12^{\frac{1}{2}}=2\sqrt{3}$.

23. Expand $(\sqrt[6]{12})^{12}$.

24. Expand $(\sqrt[9]{54\,a^2b})^3$.

25. Expand $(4\sqrt{a-b}+3\sqrt{a+b})^2$.

Expand and express result in the form $a+2\sqrt{b}$:

26. $(5+\sqrt{3})^2=25+10\sqrt{3}+3=28+2\cdot 5\sqrt{3}=28+2\sqrt{75}$.

27. $(5^{\frac{1}{2}}-3^{\frac{1}{2}})^2$.

28. $(2^{\frac{1}{2}}+3^{\frac{1}{2}})^2$.

29. $(\sqrt{5}+\sqrt{7})^2$.

30. $(\sqrt{6}-2\sqrt{3})^2$.

31. $(7+4\sqrt{3})^2$.

32. $[3+(2)^{\frac{1}{2}}]^2$.

33. $(2\cdot 2^{\frac{1}{2}}+3\cdot 3^{\frac{1}{2}})^2$.

Supply the missing term in the following trinomial squares:

34. $4+2\sqrt{12}+\,?$

35. $7+2\sqrt{14}$.

36. $14+2(14)^{\frac{1}{2}}$.

37. Extract the square root of $7+2\sqrt{12}$.

38. Extract the square root of $5+2(6)^{\frac{1}{2}}$.

Note that in squaring a binomial, one or both of whose terms are affected by the exponent $\frac{1}{2}$, the square reduces to a binomial surd if both terms of the binomial to be squared are numerical. (Compare Examples 27–36.) Also *the part* $2(\)^{\frac{1}{2}}$ *corresponds to the middle term of the examples in* §91. *In* $2(ab)^{\frac{1}{2}}$ *if* ab *can be so factored that the sum of these factors is equal to the other term, the square roots of these factors connected by the sign of the irrational term will be the square root of the binomial surd.*

39. Find the square root of $8+2(15)^{\frac{1}{2}}$.

$15=5\cdot 3$ and $5+3=8$.

Hence by the above rule,

$$\sqrt{8+2(15)^{\frac{1}{2}}}=\sqrt{5}+\sqrt{3}.$$

40. $[17+6(8)^{\frac{1}{2}}]^{\frac{1}{2}}=[17+2(72)^{\frac{1}{2}}]^{\frac{1}{2}}$

$=[17+2(8\cdot 9)^{\frac{1}{2}}]^{\frac{1}{2}}$

$=9^{\frac{1}{2}}+8^{\frac{1}{2}}$

$=9^{\frac{1}{2}}+(2^3)^{\frac{1}{2}}\cdot 2^{\frac{1}{2}}$

$=3+2\sqrt{2}$.

Find the square roots of the following binomial surds:

Remember that the coefficient of the radical must be 2.

41. $3-2\sqrt{2}$.

42. $11+2(30)^{\frac{1}{2}}$.

43. $14+6\sqrt{5}$.

44. $24+2\cdot 140^{\frac{1}{2}}$.

45. $23+2\cdot 132^{\frac{1}{2}}$.

46. $29+2\cdot 54^{\frac{1}{2}}$.

47. $55+3\sqrt{24}$.

48. $12-\sqrt{108}$.

49. $37-640^{\frac{1}{2}}$.

50. $4+(15)^{\frac{1}{2}}$.

51. $5+\sqrt{21}$.

52. $55-20\sqrt{6}$.

53. $44-4(72)^{\frac{1}{2}}$.

54. $53-\sqrt{600}$.

DIVISION OF MONOMIAL SURDS

211. $$\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}.$$

Whence, $$\frac{\sqrt[n]{ab}}{\sqrt[n]{a}}=\sqrt[n]{b}.$$

Rule:

To divide one monomial *surd* by another, reduce them, if necessary, to surds of the same degree.

Divide the expression under the radical sign in the dividend by the expression under the radical sign in the

divisor, and write the result under the common radical sign. ***The result should be reduced to its simplest form.***

1. Divide $\sqrt[3]{405}$ by $\sqrt[3]{5}$.

We have, $\dfrac{\sqrt[3]{405}}{\sqrt[3]{5}} = \sqrt[3]{\dfrac{405}{5}} = \sqrt[3]{81} = \sqrt[3]{27 \times 3} = 3\sqrt[3]{3}$.

2. Divide $\sqrt[3]{4}$ by $\sqrt{6}$.

Reducing to surds of the same degree (§ 209),

$$\frac{\sqrt[3]{4}}{\sqrt{6}} = \frac{4^{\frac{1}{3}}}{6^{\frac{1}{2}}} = \frac{(2^2)^{\frac{1}{3}}}{(2\times3)^{\frac{1}{2}}} = \frac{\sqrt[6]{2^4}}{\sqrt[6]{2^3\times3^3}} = \sqrt[6]{\frac{2^4}{2^3\times3^3}} = \sqrt[6]{\frac{2}{3^3}} = \sqrt[6]{\frac{2\times3^5}{3^6}} = \tfrac{1}{3}\sqrt[6]{54}.$$

3. Divide $\sqrt{10}$ by $\sqrt[6]{40}$.

We have, $\sqrt{10} = 10^{\frac{1}{2}} = 10^{\frac{3}{6}} = (10^3)^{\frac{1}{6}} = (5^3 \cdot 2^3)^{\frac{1}{6}}$.

Then, $\dfrac{\sqrt{10}}{\sqrt[6]{40}} = \left(\dfrac{2^3 \cdot 5^3}{2^3 \cdot 5}\right)^{\frac{1}{6}} = (5^2)^{\frac{1}{6}} = 5^{\frac{1}{3}} = \sqrt[3]{5}$.

EXERCISE 89

Divide the following:

1. $\sqrt{60}$ by $\sqrt{3}$.
2. $(72)^{\frac{1}{2}}$ by $(2)^{\frac{1}{2}}$.
3. $\sqrt{48}$ by $\sqrt{32}$.
4. $75^{\frac{1}{2}}$ by $60^{\frac{1}{2}}$.
5. $6\cdot3^{\frac{1}{2}}$ by $2\cdot3^{\frac{1}{2}}$.
6. $\sqrt[3]{32}$ by $\sqrt[3]{4}$.
7. $45^{\frac{1}{3}}$ by $9^{\frac{1}{3}}$.
8. $\sqrt{128}$ by $\sqrt{48}$.
9. $\sqrt[3]{12}$ by $\sqrt{16}$.
10. $20\sqrt{12}$ by $8\sqrt{3}$.
11. $(6\,a^2b)^{\frac{1}{5}}$ by $(96\,bc^3)^{\frac{1}{10}}$.
12. $\sqrt[5]{3\,a^3}$ by $\sqrt{2\,a}$.
13. $\sqrt{27\,x^3}$ by $\sqrt[3]{36\,x^2}$.
14. $\sqrt[3]{\frac{49}{27}}$ by $\sqrt{\frac{7}{8}}$.
15. $(\frac{52}{15})^{\frac{1}{2}}$ by $(\frac{65}{12})^{\frac{1}{2}}$.
16. $\sqrt[5]{\frac{4}{9}}$ by $\sqrt{\frac{2}{3}}$.
17. $(\frac{21}{44})^{\frac{1}{2}}$ by $(\frac{28}{33})^{\frac{1}{2}}$.
18. $(\frac{12}{15})^{\frac{1}{2}}$ by $2\cdot3^{\frac{1}{2}}$.

EVOLUTION OF SURDS

212. 1. Extract the cube root of $\sqrt[5]{27\,x^3}$.

$$\sqrt[3]{(\sqrt[5]{27\,x^3})} = (\sqrt[5]{(3\,x)^3})^{\frac{1}{3}} = [(3\,x)^{\frac{3}{5}}]^{\frac{1}{3}} = (3\,x)^{\frac{1}{5}} = \sqrt[5]{3\,x}.$$

2. Extract the fifth root of $\sqrt{6}$.

$$\sqrt[5]{(\sqrt[3]{6})} = (6^{\frac{1}{3}})^{\frac{1}{5}} = 6^{\frac{1}{15}} = \sqrt[15]{6}.$$

Then, to extract any root of a surd,

If possible, extract the required root of the expression under the radical sign; otherwise, multiply the index of the surd by the index of the required root.

If the surd has a coefficient which is not a perfect power of the degree denoted by the index of the required root, it should be introduced under the radical sign (§ 206) before applying the rule.

Thus, $\sqrt[5]{(4\sqrt{2})} = \sqrt[5]{(\sqrt{32})} = \sqrt{2}$,

or $\sqrt[5]{(4\sqrt{2})} = (4 \cdot 2^{\frac{1}{2}})^{\frac{1}{5}} = (2^2 \cdot 2^{\frac{1}{2}})^{\frac{1}{5}} = 2^{\frac{2}{5}} \cdot 2^{\frac{1}{10}} = 2^{\frac{4}{10}} \cdot 2^{\frac{1}{10}} = 2^{\frac{5}{10}} = \sqrt{2}$.

EXERCISE 90

Find the values of the following:

1. $\sqrt{(\sqrt[6]{25})}$.
2. $\sqrt[3]{(\sqrt[7]{8\,a^3b^6})}$.
3. $\sqrt{(\sqrt[3]{13})}$.
4. $\sqrt[5]{(\sqrt[4]{243\,x^5})}$.
5. $\sqrt{(\sqrt[5]{9\,a^2+12\,a+4})}$.
6. $\sqrt[3]{(\sqrt[4]{49})}$.
7. $\sqrt[4]{(81\sqrt[3]{16})}$.
8. $\sqrt[5]{(2\sqrt[3]{3\,a^2b})}$.
9. $\sqrt[4]{(16\,a^4\sqrt[4]{3\,a})}$.
10. $\sqrt[7]{(2\,x\sqrt[5]{4\,x^2})}$.
11. $\sqrt[6]{(\sqrt{343})}$.
12. $\sqrt[3]{(2\,n^2\sqrt[5]{16\,n^2})}$.

REDUCTION OF A FRACTION WHOSE DENOMINATOR IS IRRATIONAL (§ 198) TO AN EQUIVALENT FRACTION HAVING A RATIONAL DENOMINATOR

213. CASE I. *When the denominator is a monomial.*

The reduction may be effected by multiplying both terms of the fraction by a surd of the same degree as the denominator, having under its radical sign such an expression as will make the denominator of the resulting fraction rational.

Ex. Reduce $\frac{5}{\sqrt[3]{3\,a^2}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $\sqrt[3]{9\,a}$, we have

$$\frac{5}{\sqrt[3]{3\,a^2}} = \frac{5\sqrt[3]{9\,a}}{\sqrt[3]{3\,a^2}\sqrt[3]{9\,a}} = \frac{5\sqrt[3]{9\,a}}{\sqrt[3]{27\,a^3}} = \frac{5\sqrt[3]{9\,a}}{3\,a}.$$

EXERCISE 91

Reduce each of the following to an equivalent fraction having a rational denominator:

1. $\frac{2}{\sqrt{3}}$.
2. $\frac{1}{\sqrt{2}}$.
3. $\sqrt{\frac{5}{2}}$.
4. $\frac{5}{\sqrt{7}}$.
5. $\frac{3}{\sqrt[3]{4}}$.
6. $\frac{a^2}{\sqrt[3]{9a^2}}$.
7. $\frac{9b}{\sqrt[3]{36b}}$.
8. $\frac{9}{\sqrt[5]{27}}$.

214. CASE II. *When the denominator is a binomial containing only surds of the second degree.*

1. Reduce $\frac{5-\sqrt{2}}{5+\sqrt{2}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $5-\sqrt{2}$ ($5-\sqrt{2}$ is called the conjugate of $5+\sqrt{2}$), we have

$$\frac{5-\sqrt{2}}{5+\sqrt{2}}=\frac{(5-\sqrt{2})^2}{(5+\sqrt{2})(5-\sqrt{2})}=\frac{25-10\sqrt{2}+2}{25-2}\ (\S\S\ 89, 91)\ =\frac{27-10\sqrt{2}}{23}.$$

2. Reduce $\frac{3\sqrt{a}-2\sqrt{a-b}}{2\sqrt{a}-3\sqrt{a-b}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms by $2\sqrt{a}+3\sqrt{a-b}$,

$$\frac{3\sqrt{a}-2\sqrt{a-b}}{2\sqrt{a}-3\sqrt{a-b}}=\frac{(3\sqrt{a}-2\sqrt{a-b})(2\sqrt{a}+3\sqrt{a-b})}{(2\sqrt{a}-3\sqrt{a-b})(2\sqrt{a}+3\sqrt{a-b})}$$
$$=\frac{6a+5\sqrt{a}\sqrt{a-b}-6(a-b)}{4a-9(a-b)}=\frac{6b+5\sqrt{a^2-ab}}{9b-5a}.$$

RULE: — **Multiply both numerator and denominator of the fraction by the denominator with the sign between its terms changed.**

EXERCISE 92

Reduce each of the following to an equivalent fraction having a rational denominator:

1. $\frac{3}{\sqrt{3}+\sqrt{2}}$.
2. $\frac{8}{3-(5)^{\frac{1}{2}}}$.
3. $\frac{4-(2)^{\frac{1}{2}}}{4+(2)^{\frac{1}{2}}}$.

4. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$. 5. $\dfrac{\sqrt{10}-6\sqrt{2}}{\sqrt{10}+3\sqrt{2}}$. 6. $\dfrac{(x^2+a^2)^{\frac{1}{2}}+a}{(x^2+a^2)^{\frac{1}{2}}-a}$.

7. $1+\dfrac{a^2}{\sqrt{x^2-a^2}}$.

8. $\dfrac{x^{\frac{1}{2}}-(x+y)^{\frac{1}{2}}}{x^{\frac{1}{2}}+(x+y)^{\frac{1}{2}}}$.

9. $\dfrac{2+\sqrt{\dfrac{2}{x}}}{2-\sqrt{\dfrac{2}{x}}}$.

10. $\dfrac{3-(a-3)^{\frac{1}{2}}}{3+(a-3)^{\frac{1}{2}}}$. 11. $\dfrac{5\cdot 2^{\frac{1}{2}}+6^{\frac{1}{2}}}{3\cdot 2^{\frac{1}{2}}-6^{\frac{1}{2}}}$. 12. $\dfrac{\sqrt{x-2}+1}{\sqrt{x-2}+2}$.

Add the following fractions:

(The common denominator is more readily found if the denominators of the fractions are first rationalized.)

13. $\dfrac{2}{3^{\frac{1}{2}}+2^{\frac{1}{2}}}+\dfrac{3}{3^{\frac{1}{2}}-2^{\frac{1}{2}}}$.

14. $\dfrac{2\sqrt{6}+1}{\sqrt{3}+2}+\dfrac{5+\sqrt{6}}{\sqrt{12}}$.

15. $\dfrac{a+b^{\frac{1}{2}}}{a-2\,b^{\frac{1}{2}}}-\dfrac{a-b^{\frac{1}{2}}}{a+b^{\frac{1}{2}}}$.

16. $\dfrac{(x+1)^{\frac{1}{2}}+2}{(x+1)^{\frac{1}{2}}-2}+\dfrac{3\sqrt{2}}{(2\,x+2)^{\frac{1}{2}}}$.

215. The approximate value of a fraction whose denominator is irrational may be conveniently found by reducing it to an equivalent fraction with a rational denominator.

Ex. **Find the approximate value of $\dfrac{1}{2-\sqrt{2}}$ to three places of decimals.**

$$\frac{1}{2-\sqrt{2}}=\frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})}=\frac{2+\sqrt{2}}{4-2}=\frac{2+1.414\cdots}{2}=1.707\cdots.$$

The $\sqrt{2}$ and the $\sqrt{3}$ are important values and are of frequent occurrence in mathematical investigation.

EXERCISE 93

Find the values of the following to three places of decimals:

1. $\dfrac{3}{\sqrt{5}}$. 2. $\dfrac{1}{\sqrt{2}}$. 3. $\dfrac{1}{\sqrt{3}}$.

4. $\frac{2}{3-\sqrt{3}}$. 6. $\frac{2}{\sqrt{5}-1}$. 8. $\frac{3}{\sqrt{10}}$.

5. $\frac{1}{5+2\sqrt{7}}$. 7. $\frac{5}{2\sqrt{3}-4}$. 9. $\frac{1}{\sqrt{6}-2}$.

216. Important Property of Quadratic Surds (§ 201). I. *A quadratic surd cannot equal the sum of a rational expression and a quadratic surd.*

For, if possible, let $\sqrt{a}=b+\sqrt{c}$,

where b is a rational expression, and $\sqrt{a}$ and $\sqrt{c}$ quadratic surds.

Squaring both members, $a=b^2+2b\sqrt{c}+c$,

or, $2b\sqrt{c}=a-b^2-c$.

Whence, $\sqrt{c}=\frac{a-b^2-c}{2b}$.

That is, a quadratic surd equal to a rational expression.

But this is impossible; whence, $\sqrt{a}$ cannot equal $b+\sqrt{c}$.

II. *If* $a+\sqrt{b}=c+\sqrt{d}$, *where* a *and* c *are rational expressions, and* $\sqrt{b}$ *and* $\sqrt{d}$ *quadratic surds, then*

$$a=c, \text{ and } \sqrt{b}=\sqrt{d}.$$

If a does not equal c, let $a=c+x$; then, x is rational.

Substituting this value in the given equation,

$$c+x+\sqrt{b}=c+\sqrt{d}, \text{ or } x+\sqrt{b}=\sqrt{d}.$$

But this is impossible by § 216.

Then, $a=c$, and therefore $\sqrt{b}=\sqrt{d}$.

That is, an equation of the form $a+\sqrt{b}=c+\sqrt{d}$ may be written as two equations,

$$a=c, \quad b=d.$$

217. Solution of Equations having the Unknown Numbers under Radical Signs.

1. Solve the equation $(x^2-5)^{\frac{1}{2}}-x=-1$.

Transposing $-x$, $(x^2-5)^{\frac{1}{2}}=x-1.$

Squaring both members, $x^2-5=x^2-2x+1.$

Transposing, $2x=6$; whence, $x=3.$

(Substituting 3 for x in the given first member, and taking the positive value of the square root, the first member becomes

$$(9-5)^{\frac{1}{2}}-3=2-3=-1;$$

which shows that the solution $x=3$ is correct.)

Where no sign occurs before a radical the positive sign is understood.

Also, in equations of the type, $\sqrt{x+3}+\sqrt{x^2+9}-\sqrt{x-3}=\sqrt{x}$, usage requires that we regard only the given sign before the radical rather than the double sign that naturally belongs to a radical.

RULE: — Transpose the terms of the equation so that a surd term may stand alone in one member; then raise both members to a power of the same degree as the surd.

If surd terms still remain, repeat the operation.

The equation should be simplified as much as possible before performing the involution.

2. Solve the equation $\sqrt{2x-1}+\sqrt{2x+6}=7.$

Transposing $\sqrt{2x-1}$, $\sqrt{2x+6}=7-\sqrt{2x-1}.$

Squaring, $2x+6=49-14\sqrt{2x-1}+2x-1.$

Transposing, $14\sqrt{2x-1}=42$, or $\sqrt{2x-1}=3.$

Squaring, $2x-1=9$; whence, $x=5.$

Substitute $x=5$ in the given equation to verify the result.

EXERCISE 94

Solve the following equations, verifying each result:

(Any radical may be changed to a form with fractional exponents and the exponent form is often more easily solved.)

1. $(2x+1)^{\frac{1}{2}}-5=0.$
2. $\sqrt{2x+7}+5=8.$
3. $(4t^2-19)^{\frac{1}{2}}=2t-1.$
4. $\sqrt{u^2-11}+u=11.$
5. $\dfrac{2}{(z+8)^{\frac{1}{2}}}-\dfrac{2z}{(2-2z)^{\frac{1}{2}}}=(2-2z)^{\frac{1}{2}}.$
6. $\sqrt{5m-24}+4=\sqrt{5m}.$
7. $(v)^{\frac{1}{2}}+(v-3)^{\frac{1}{2}}=3.$

8. $\dfrac{(3x+24)^{\frac{1}{2}}-(3x)^{\frac{1}{2}}}{(3x+24)^{\frac{1}{2}}+(3x)^{\frac{1}{2}}}=\dfrac{1}{2}$ (§ 147).

9. $\sqrt{u-3}-\sqrt{u+21}=-2\sqrt{u}$.

10. $(8x^3+36x^2)^{\frac{1}{3}}-3=2x$.

11. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=2$. (Rationalize denominator, or use § 147.)

12. $(t^2-5t-2)^{\frac{1}{2}}+(t^2+3t+6)^{\frac{1}{2}}=4$.

13. $\sqrt{x+15}-\sqrt{x+3}=2\sqrt{x}$.

14. $\dfrac{(2x+8)^{\frac{1}{2}}+(2x)^{\frac{1}{2}}}{(2x+8)^{\frac{1}{2}}-(2x)^{\frac{1}{2}}}=2$.

15. $\sqrt{x-2a}-\sqrt{x-6a}=2\sqrt{x-5a}$.

IMAGINARY NUMBERS

218. If a number involves an indicated even root of a negative number it is called *imaginary*. Such numbers depend upon a new unit, $\sqrt{-1}$ or $(-1)^{\frac{1}{2}}$; as $\sqrt{-2}$, $\sqrt[4]{-3}$.

In contradistinction, rational and irrational numbers (§ 199) are called *real* numbers.

219. An imaginary number of the form $\sqrt{-a}$ is called a *pure imaginary* number, and the sum of a real and an imaginary is called a *complex* number; as $a+b\sqrt{-1}$.

220. Meaning of a Pure Imaginary Number.

If $\sqrt{a}$ is *real* (§ 218), we define $\sqrt{a}$ as an expression such that, when raised to the second power, the result is a (§ 165).

To find what meaning to attach to a pure imaginary number, we assume the above principle to hold when $\sqrt{a}$ is imaginary.

Thus, $\sqrt{-2}$ means an expression such that, when raised to the second power, the result is -2; that is, $(\sqrt{-2})^2$ or $(-2^{\frac{1}{2}})^2=-2$.

In like manner, $(\sqrt{-1})^2=(-1^{\frac{1}{2}})^2=-1$; etc.

OPERATIONS WITH IMAGINARY NUMBERS

221. By § 220, $(\sqrt{-5})^2=(-5^{\frac{1}{2}})^2=-5.$ (1)

Also, $(\sqrt{5}\sqrt{-1})^2=(\sqrt{5})^2(\sqrt{-1})^2=5(-1)=-5,$ (2)

or $(\sqrt{-5})^2=(5^{\frac{1}{2}})^2\cdot(-1^{\frac{1}{2}})^2=5(-1)=-5.$

From (1) and (2), $(\sqrt{-5})^2=(\sqrt{5}\sqrt{-1})^2.$

Whence, $\sqrt{-5}=\sqrt{5}\sqrt{-1}$, or $5^{\frac{1}{2}}(-1)^{\frac{1}{2}}.$

Then, *every imaginary square root can be expressed as the product of a real number by* $\sqrt{-1}$. *It is advisable to reduce every imaginary to this form before performing the indicated operations.*

$\sqrt{-1}$ is called the *imaginary unit;* it is often represented by i.

222. Addition and Subtraction of Imaginary Numbers.

Pure imaginary numbers may be added and subtracted in the same manner as surds.

1. Add $\sqrt{-4}$ and $\sqrt{-36}$.

By § 221, $\sqrt{-4}+\sqrt{-36}=2(-1)^{\frac{1}{2}}+6(-1)^{\frac{1}{2}}=8(-1)^{\frac{1}{2}}.$

2. Subtract $3-\sqrt{-9}$ from $1+\sqrt{-16}$.

In adding or subtracting complex numbers, we assume that the rules for adding or subtracting real numbers may be applied without change.

Then, $1+\sqrt{-16}-(3-\sqrt{-9})=1+4\sqrt{-1}-3+3\sqrt{-1}$

$=-2+7\sqrt{-1}.$

EXERCISE 95

Simplify the following:

1. $(-9)^{\frac{1}{2}}+(-25)^{\frac{1}{2}}.$
2. $\sqrt{-5}+\sqrt{-45}.$
3. $\sqrt{-18}-\sqrt{-8}.$
4. $\sqrt{-75}+\sqrt{-48}.$
5. $\sqrt{-(x+2)^2}-\sqrt{-x^2}.$
6. $(-x^2)^{\frac{1}{2}}+(-y^2)^{\frac{1}{2}}+(-z^2)^{\frac{1}{2}}.$
7. $3\sqrt{-36}+2\sqrt{-144}+\sqrt{-81}.$
8. $2(-16)^{\frac{1}{2}}-5(-49)^{\frac{1}{2}}-8(-121)^{\frac{1}{2}}.$
9. $\sqrt{-16\,x^2}-\sqrt{-9\,x^2}-\sqrt{-4\,x^2}.$

10. $\sqrt{-4a^2-4ab-b^2}-\sqrt{-9a^2+6ab-b^2}$.

11. Add $2+(-3)^{\frac{1}{2}}$ to $5+(-27)^{\frac{1}{3}}$.

12. Add $6-\sqrt{-64}$ to $1-\sqrt{-49}$.

13. From $2+\sqrt{-9}$ take $8-\sqrt{-25}$.

223. Positive Integral Powers of $\sqrt{-1}$ or $-1^{\frac{1}{2}}$.

By § 220, $(-1^{\frac{1}{2}})^2=(-1)^{\frac{1}{2}}\cdot(-1)^{\frac{1}{2}}=-1$. (By adding exponents.)

Then, $(\sqrt{-1})^2=(-1^{\frac{1}{2}})^2\cdot(-1)^{\frac{1}{2}}=-1\cdot(-1)^{\frac{1}{2}}=-\sqrt{-1}$;

$(\sqrt{-1})^4=(-1^{\frac{1}{2}})^2\cdot(-1^{\frac{1}{2}})^2=-1\cdot-1=1$;

$(\sqrt{-1})^5=(-1^{\frac{1}{2}})^4\cdot(-1)^{\frac{1}{2}}=1\cdot-1^{\frac{1}{2}}=\sqrt{-1}$, etc.

Thus, the first four positive integral powers of $\sqrt{-1}$ are $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1; and for higher powers these terms recur in the same order, the sixth power being like the second, etc.

224. Multiplication of Imaginary Numbers.

The product of two or more imaginary square roots can be obtained by aid of the principles of §§ 221 and 223.

1. Multiply $\sqrt{-2}$ by $\sqrt{-3}$.

By § 221, $-2^{\frac{1}{2}}\cdot-3^{\frac{1}{2}}=2^{\frac{1}{2}}\cdot-1^{\frac{1}{2}}\cdot3^{\frac{1}{2}}\cdot-1^{\frac{1}{2}}$

$=2^{\frac{1}{2}}\cdot3^{\frac{1}{2}}\cdot(-1^{\frac{1}{2}})^2=6^{\frac{1}{2}}(-1)$ (§ 223) $=-\sqrt{6}$.

2. Find the product of $\sqrt{-9}$, $\sqrt{-16}$, and $\sqrt{-25}$.

$\sqrt{-9}\times\sqrt{-16}\times\sqrt{-25}=3\sqrt{-1}\times4\sqrt{-1}\times5\sqrt{-1}$

$=60(\sqrt{-1})^3=60(-\sqrt{-1})$ (§ 223) $=-60\sqrt{-1}$.

3. Multiply $2+5\sqrt{-5}$ by $4-3\sqrt{-5}$.

In multiplying complex numbers, we assume that the rules for multiplying real numbers may be applied without change.

$$\begin{array}{l} 2+\ 5\sqrt{5}\sqrt{-1} \\ \underline{4-\ 3\sqrt{5}\sqrt{-1}} \\ 8+20\sqrt{5}\sqrt{-1} \\ \underline{\quad -\ 6\sqrt{5}\sqrt{-1}-15(5)(-1)} \\ 8+14\sqrt{-5}\quad\ +75=83+14\sqrt{-5} \end{array}$$

4. Expand $(\sqrt{-5}+2\sqrt{-3})^2$ by the rule of § 91.

$$(-5^{\frac{1}{2}}+2\cdot-3^{\frac{1}{2}})^2=(5^{\frac{1}{2}}\cdot-1^{\frac{1}{2}}+2\cdot3^{\frac{1}{2}}\cdot-1^{\frac{1}{2}})^2$$
$$=(5^{\frac{1}{2}})^2\cdot(-1^{\frac{1}{2}})^2+4\cdot5^{\frac{1}{2}}\cdot-1^{\frac{1}{2}}\cdot3^{\frac{1}{2}}\cdot-1^{\frac{1}{2}}+2^2\cdot(3^{\frac{1}{2}})^2\cdot(-1^{\frac{1}{2}})^2$$
$$=-5+4\cdot15^{\frac{1}{2}}\cdot(-1^{\frac{1}{2}})^2+4\cdot-3$$
$$=-5-4\cdot15^{\frac{1}{2}}-12=-17-4\sqrt{15}.$$

EXERCISE 96

Multiply the following:

1. $(-4)^{\frac{1}{2}}$ by $(-9)^{\frac{1}{2}}$.
2. $(-36)^{\frac{1}{2}}$ by $(-16)^{\frac{1}{2}}$.
3. $\sqrt{-8}$ by $\sqrt{-6}$.
4. $(-196\,a^2)^{\frac{1}{2}}$ by $(-144\,a^2)^{\frac{1}{2}}$.
5. $\sqrt{-16}$ by $\sqrt{-64}$.
6. $(-9)^{\frac{1}{2}}$ by $(-18)^{\frac{1}{2}}$.
7. $3+(-3)^{\frac{1}{2}}$ by $2+(-2)^{\frac{1}{2}}$.
8. $5+4\sqrt{-1}$ by $2-\sqrt{-3}$.
9. $3\sqrt{-x}+2\sqrt{-y}$ by $2\sqrt{-x}+3\sqrt{-y}$.
10. $8\sqrt{-7}$, $5\sqrt{-5}$, $-3\sqrt{-4}$, and $2\sqrt{-6}$.
11. $\sqrt{-16}$, $\sqrt{-49}$, $\sqrt{-64}$, and $\sqrt{-100}$.

Expand the following by inspection:

12. $[2+(-3)^{\frac{1}{2}}]^2$.
13. $(5-\sqrt{-7})^2$.
14. $(5\sqrt{-6}+3\sqrt{-3})^2$.
15. $[6+4(-3)^{\frac{1}{2}}][6-4(-3)^{\frac{1}{2}}]$.
16. $(\sqrt{-3\,x}+y)(\sqrt{-3\,x}-y)$.
17. $[(-5\,x)^{\frac{1}{2}}+7^{\frac{1}{2}}][(-5\,x)^{\frac{1}{2}}+5^{\frac{1}{2}}]$.
18. $(8\sqrt{-2}+3\sqrt{-5})(8\sqrt{-2}-3\sqrt{-5})$.
19. $(a+b\sqrt{-1})(a-b\sqrt{-1})$.
20. Add $(a+b\sqrt{-1})$ to $(a-b\sqrt{-1})$.

$a+b\sqrt{-1}$ and $a-b\sqrt{-1}$ are called CONJUGATE IMAGINARIES. Note that their sum and their product are *real*.

225. Division of Imaginary Numbers.

1. Divide $\sqrt{-40}$ by $\sqrt{-5}$.

By § 221, $\frac{-40^{\frac{1}{2}}}{-5^{\frac{1}{2}}} = \frac{40^{\frac{1}{2}} \cdot -1^{\frac{1}{2}}}{5^{\frac{1}{2}} \cdot -1^{\frac{1}{2}}} = \frac{40^{\frac{1}{2}}}{5^{\frac{1}{2}}} = 8^{\frac{1}{2}} = 2\sqrt{2}$.

2. Divide $\sqrt{15}$ by $\sqrt{-3}$.

$$\frac{\sqrt{15}}{\sqrt{-3}} = \frac{-\sqrt{15}(-1)}{\sqrt{3}\sqrt{-1}} = \frac{-\sqrt{15}(\sqrt{-1})^2}{\sqrt{3}\sqrt{-1}} \text{ (§ 223)} = -\sqrt{5}\sqrt{-1} = -\sqrt{-5}.$$

3. Reduce $\frac{\sqrt{3}-\sqrt{-2}}{\sqrt{3}+\sqrt{-2}}$ to an equivalent fraction having a real denominator.

Multiply both numerator and denominator of the fraction by the conjugate of the denominator, i. e., $\sqrt{3}-\sqrt{-2}$, that is, the denominator with the sign between its terms changed.

$$\frac{\sqrt{3}-\sqrt{-2}}{\sqrt{3}+\sqrt{-2}} = \frac{(\sqrt{3}-\sqrt{-2})^2}{(\sqrt{3})^2-(\sqrt{-2})^2} \text{ (§ 89)}$$

$$= \frac{(\sqrt{3})^2-2\sqrt{3}\sqrt{-2}+(\sqrt{-2})^2}{3-(-2)} \text{ (§ 91)}$$

$$= \frac{3-2\sqrt{-6}-2}{3+2} = \frac{1-2\sqrt{-6}}{5}.$$

EXERCISE 97

Divide the following:

1. $-20^{\frac{1}{2}}$ by $-5^{\frac{1}{2}}$.
2. $\sqrt{-18}$ by $\sqrt{-3}$.
3. $-36^{\frac{1}{2}}$ by $-12^{\frac{1}{2}}$.
4. $-(-12\,a^2b)^{\frac{1}{2}}$ by $(3\,ab)^{\frac{1}{2}}$.
5. $-\sqrt{32}$ by $-\sqrt{-8}$.
6. $(180)^{\frac{1}{2}}$ by $-(10)^{\frac{1}{2}}$.
7. $-\sqrt[4]{-18}$ by $\sqrt[4]{-2}$.
8. $-\sqrt{a}$ by $\sqrt{-a^2}$.

Reduce each of the following to an equivalent fraction having a real denominator:

9. $\frac{2}{2-\sqrt{-3}}$.
10. $\frac{2+(-3)^{\frac{1}{2}}}{2-(-3)^{\frac{1}{2}}}$.
11. $\frac{3\sqrt{-a}+2\sqrt{-b}}{3\sqrt{-a}-2\sqrt{-b}}$.
12. $\frac{2\sqrt{-5}-5}{2\sqrt{-5}+5}$.

XIV. QUADRATIC EQUATIONS

226. A **Quadratic Equation** is an equation of the second degree (§ 75), with one or more unknown numbers.

A **Pure Quadratic Equation** is a quadratic equation involving only the square of the unknown number; as, $2x^2=5$.

An **Affected Quadratic Equation** is a quadratic equation involving both the square and the first power of the unknown number; as, $2x^2-3x-5=0$.

In § 103, we showed how to solve quadratic equations of the forms

$$ax^2+bx=0,\ ax^2+c=0,\ x^2+ax+b=0, \text{ and } ax^2+bx+c=0,$$

when the first members could be resolved into factors.

PURE QUADRATIC EQUATIONS

227. Let it be required to solve the equation

$$x^2-4=0,$$

or $$x^2=4.$$

Taking the square root of each member, we have

$$\pm x=\pm 2;$$

for the square root of a number may be either + or − (§ 168).

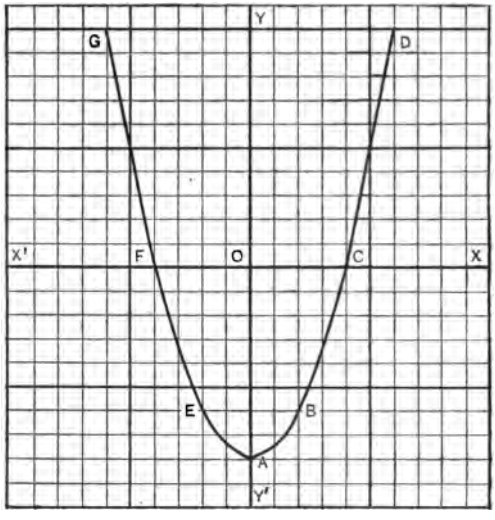
But the equations $-x=2$ and $x=-2$ are the same as $x=-2$ and $x=2$, respectively, with all signs changed.

We then get all the values of x by equating the *positive* square root of the first member to the $\pm$ square root of the second.

The graph of a quadratic expression, with one unknown number, x, may be found by putting y equal to the expression, and finding the graph of the resulting equation as in § 151.

In the equation $x^2-4=0$ placing $y=x^2-4$,

that is, substituting $y=0$, and finding values for y by assigning values 0, 1, 2, etc., to x, we have



x	y	
0	−4	(A)
1	−3	(B)
2	0	(C)
3	5	(D)
4	12	
−1	−3	(E)
−2	−0	(F)
−3	5	(G)
−4	12	

Connecting these points (A), (B), (C), etc., we find, that they form a smooth curve, that the lowest point of the curve is on the y-axis, that the curve crosses the x-axis at ± 2. That is, the curve $x^2-4=0$ crosses the line $y=0$ (the x-axis) in two points and that these points of intersection (2, 0), (−2, 0) correspond to the algebraic solution of $x^2-4=0$ in § 227.

In general, it will be found that the graph of every equation of the second degree (§ 75) in two variables (unknown numbers) is a curve. The above geometrical picture shows in a graphical way why a quadratic equation has two roots.

EXERCISE 98

Find the graph of:

1. $x^2-9=0$. 2. $x^2-16=0$. 3. $x^2-25=0$.

228. A pure quadratic equation may be solved by reducing it, if necessary, to the form $x^2=a$, and then equating x to $\pm$ the square root of a (§ 227).

1. Solve the equation $3x^2+7=\frac{5x^2}{4}+35$.

Clearing of fractions, $12x^2+28=5x^2+140$.
Transposing, and uniting terms, $7x^2=112$, or $x^2=16$.
Equating x to the $\pm$ square root of 16, $x=\pm 4$.
Verify by substituting $x=\pm 4$ in the given equation.

2. Solve the equation $7x^2-5=5x^2-13$.

Transposing, and uniting terms, $2x^2=-8$, or $x^2=-4$.

Equating x to the $\pm$ square root of -4, $x=\pm\sqrt{-4}$

$=\pm 2\sqrt{-1}$ (§ 221).

In this case, both values of x are *imaginary* (§ 219); it is impossible to find a real value of x which will satisfy the given equation.

Verify by substituting in the given equation.

Make a graph of $x^2=-4$.

In solving fractional quadratic equations, any solution which does not satisfy the given equation must be rejected.

Thus, let it be required to solve the equation

$$\frac{x^2-7}{x^2+x-2}=\frac{1}{x+2}-\frac{1}{x-1}.$$

Multiplying both members by $(x+2)(x-1)$, or x^2+x-2,

$$x^2-7=x-1-x-2, \text{ or } x^2=4.$$

Extracting square roots, $x=\pm 2$.

The solution $x=-2$ does not satisfy the given equation; the only solution is $x=2$.

EXERCISE 99

Solve the following equations and verify each result:

1. $8v^2-24=7v^2+25$.

2. $\dfrac{5}{3x^2}-\dfrac{4}{5x^2}=\dfrac{13}{60}$.

3. $3(2t+4)^2+4(3t-2)^2=256$.

4. $\dfrac{4x}{3}+\dfrac{2}{3x}=\dfrac{7x}{12}+\dfrac{2}{x}$.

5. $\dfrac{2x^2+4}{5}-\dfrac{3x^2-7}{3}=\dfrac{11}{15}$.

6. $\dfrac{3x^2-4}{12}-\dfrac{2x^2+2}{5x^2+4}=\dfrac{x^2-3}{4}$. (§ 132, Ex 3.)

7. $\sqrt{10+u}-\sqrt{10-u}=2$.

8. $\dfrac{4v^2+3}{7}-\dfrac{8v^2-1}{2}=\dfrac{1}{14}$.

9. $\dfrac{3a}{t-5b}-\dfrac{t+5b}{3a+10b}=0$. Solve for t.

10. $\dfrac{4x^2-1}{4}+\dfrac{3x^2-4}{3}=\dfrac{2x^2-3}{2}$.

11. $\frac{t+a}{t-a}+\frac{t-a}{t+a}=\frac{a+c}{a-c}+\frac{a-c}{a+c}$. (§ 133, Exs. 1, 2.)

12. $\sqrt{x^2+2}=-x-\frac{4}{\sqrt{x^2+2}}$.

The following equations occur in the study of physics.

Solve in the first six equations for the number which appears to the second power.

13. $S=\frac{1}{2}gt^2$.

14. $E=\frac{1}{2}mv^2$.

15. $F=\frac{mM}{d^2}$.

16. $H=C^2Rt$.

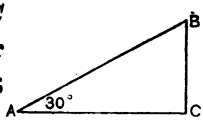
17. $f=\frac{mv^2}{R}$.

18. $R=\frac{kl}{D^2}$.

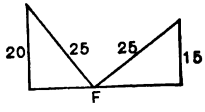
19. If the square of a certain number divided by 4 is added to twice the square of the number divided by 32, the sum is 20; find the number.

20. One number is five times another, and the difference of their squares is 216; find the numbers.

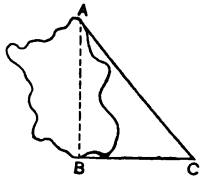
21. If one angle of the right triangle ABC is 30°, the hypothenuse is twice the shorter side. The side opposite angle B is $10\sqrt{3}$; find CB and explain your negative root.



22. A ladder 25 feet long standing in a court, will reach a window on one side of the court 20 feet from the ground. If turned on its foot as an axis it will reach a window in the opposite wall 15 feet from the ground. Find distance across the court and explain your negative roots.

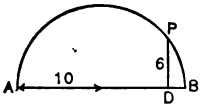


23. Two camps, A and B, are at opposite sides of a lake. In order to find the distance between them, a line BC was measured at right angles to AB. BC was found to be 441 feet. AC was measured and found to be 735 feet. Find AB.



24. In a semicircle, if the perpendicular DP be dropped from a point P in the circumference to the diameter AB, DP

is a mean proportional between the segments AD and DB. If the perpendicular DP is 6, find AD and DB, the radius of the circle being 10. If DP is 10 and the radius 6, what effect does it have on your solution? Draw the figure.



25. When a body falls from rest from any point above the earth's surface, the distance, S, which it traverses in any number of seconds, t, is found to be given by the equation

$$S=\tfrac{1}{2}gt^2,$$

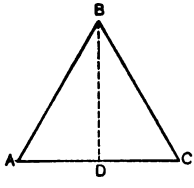
in which g represents the velocity which the body acquires in one second; $g=32.15$ feet, or 980 centimeters.

A stone fell from a balloon a mile high; how much time elapsed before it reached the earth?

26. In the equation $t=\pi\sqrt{\frac{l}{g}}$, t represents the time required by a pendulum to make one vibration, l represents the length of the pendulum, and g is the same as in Problem 25. Find the length of a pendulum which beats seconds.

27. If a pendulum which beats seconds is found to be 99.3 centimeters long, find from the above equation the value of g.

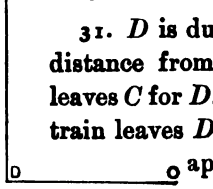
28. The area of an equilateral triangle ABC is $16\sqrt{3}$. Find the altitude DB.



29. Two balloons start at the same time from St. Louis on a long-distance race. One strikes a northwest current carrying it 30 miles an hour; the other strikes a southwest current carrying it 25 miles an hour. At the end of the second hour each balloon is one mile from the earth. How far apart are they?

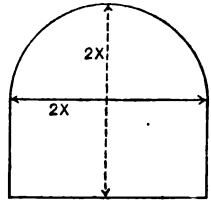
30. Two automobiles start from A at the same time, one going north at 18 miles an hour, the other going east at 15

miles an hour. How far apart are they at the end of the first hour?



31. D is due west of C, A is due north of D, and the distance from C to D is 84 miles. At 2 P. M. a train leaves C for D, running 40 miles an hour. At 2:30 P. M. a train leaves D for A, running 44 miles an hour. How far apart are they at 3 P. M.?

32. A window in the form of a rectangle surmounted by a semicircle is found to admit the most light when its height and width are equal. If the area of this window is 32.1372, find the width.



AFFECTED QUADRATIC EQUATIONS

What must be added to x^2+4x to form a perfect trinomial square? (Exercise 30.) What must be added to x^2+10x to form a perfect trinomial square?

229. First Method of Completing the Square.

By transposing the terms involving x to the first member, and all other terms to the second, and then dividing both members by the coefficient of x^2, any affected quadratic equation can be reduced to the form $x^2+px=q$.

We then add to both members such an expression as will make the first member a trinomial perfect square (Exercise 30); an operation which is termed *completing the square.*

Ex. Solve the equation $x^2+3x=4$.

A trinomial is a perfect square when its first and third terms are perfect squares and positive, and its second term plus or minus twice the product of their square roots (Exercise 30).

Then, *the square root of the third term is equal to the second term divided by twice the square root of the first.*

Hence, the *square root* of the expression which must be added to x^2+3x to make it a perfect square is $3x \div 2x$, or $\frac{3}{2}$.

Adding to both members the square of $\frac{3}{2}$, we have

$$x^2+3x+(\tfrac{3}{2})^2=4+\tfrac{9}{4}=\tfrac{25}{4}.$$

Equating the square root of the first member to the $\pm$ square root of the second (compare § 227), we have

$$x+\tfrac{3}{2}=\pm\tfrac{5}{2}.$$

Transposing $\frac{3}{2}$, $\quad x=-\tfrac{3}{2}+\tfrac{5}{2}$ or $-\tfrac{3}{2}-\tfrac{5}{2}=1$ or -4.

Rule:

Reduce the equation to the form $x^2+px=q$.

Complete the square, by adding to both members the square of one-half the coefficient of x.

Equate the square root of the first member to the $\pm$ square root of the second, and solve the linear equations thus formed.

230. 1. Solve the equation $3x^2-8x=-4$.

Dividing by 3, $\quad x^2-\dfrac{8x}{3}=-\dfrac{4}{3}$,

which is in the form $\quad x^2+px=q$.

Adding to both members the square of $\frac{4}{3}$, we have

$$x^2-\frac{8x}{3}+\left(\frac{4}{3}\right)^2=-\frac{4}{3}+\frac{16}{9}=\frac{4}{9}.$$

Equating the square root of the first member to the $\pm$ square root of $\frac{4}{9}$, $\quad x-\dfrac{4}{3}=\pm\dfrac{2}{3}$.

Transposing $-\frac{4}{3}$, $\quad x=\dfrac{4}{3}\pm\dfrac{2}{3}=2$ or $\dfrac{2}{3}$.

If the *coefficient of x^2 is negative*, the *sign* of *each term* must be changed.

2. Solve the equation $-9x^2-21x=10$.

Dividing by -9, $\quad x^2+\dfrac{7x}{3}=-\dfrac{10}{9}$.

Adding to both members the square of $\frac{7}{6}$,

$$x^2+\frac{7x}{3}+\left(\frac{7}{6}\right)^2=-\frac{10}{9}+\frac{49}{36}=\frac{9}{36}.$$

Extracting square roots, $x+\frac{7}{6}=\pm\frac{3}{6}$.

Then, $x=-\frac{7}{6}\pm\frac{3}{6}=-\frac{2}{3}$ or $-\frac{5}{3}$.

EXERCISE 100

Solve the following equations and verify each result:

1. $t^2+4t=32$.
2. $u^2-u=6$.
3. $v^2-8v=-12$.
4. $m^2-2m=15$.
5. $4x^2+4x=3$.
6. $12k^2-k=1$.
7. $9t^2-3t=2$.
8. $9t^2-9t=-2$.
9. $9t^2+9t=4$.
10. $16z^2-8z=15$.

231. The graphs of affected quadratic equations can be readily constructed by the method used in § 227.

Construct the graph or geometrical picture of

$$x^2-x-6=0. \quad (1)$$

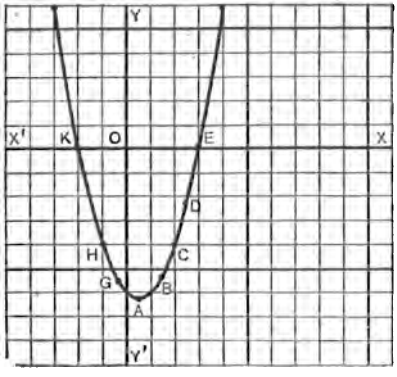
Placing the first member of the equation equal to y, we have

$$x^2-x-6=y. \quad (2)$$

Assigning values to x, we obtain corresponding values of y. For example,

Substituting $x=0$ in (2), we have $y=-6$,

Substituting $x=2$ in (2), we have $y=-4$, etc.



$x^2-x-6=y$

x	y	
0	-6	
$\frac{1}{2}$	$-6\frac{1}{4}$	(A)
1	-6	
$\frac{3}{2}$	$-5\frac{1}{4}$	(B)
2	-4	(C)
$\frac{5}{2}$	$-2\frac{1}{4}$	(D)
3	0	(E)
4	6	
$-\frac{1}{2}$	$-5\frac{1}{4}$	(G)
-1	-4	(H)
-2	0	(K)
-3	6	

Solving $x^2-x-6=0$

or $(x-3)(x+2)=0,$

we have, $x=3$ or $-2.$

In the graph of this affected quadratic equation note

(*a*) that the lowest point of the curve is not on the y-axis;

(*b*) that the curve crosses the x-axis in two points ($x=3$, $x=-2$) corresponding to the algebraic solution.

The graph of every equation of the form $x^2+px=q$ or $ax^2+bx+c=0$ is a curve of the above form and is called a *parabola.*

EXERCISE 101

Construct the graph of the following equations and compare the points of intersection with the algebraic solution:

1. $x^2-x-2=0.$
2. $x^2-8x+15=0.$
3. $x^2+6x=-8.$
4. $8x^2+6x=-1.$
5. $8x^2-2x=1.$
6. $3x^2-17x-6=0.$

232. Second Method of Completing the Square.

Every affected quadratic equation can be reduced to the form $ax^2+bx+c=0$, or $ax^2+bx=-c.$

Multiplying both members by $4a$, we have

$$4a^2x^2+4abx=-4ac.$$

We complete the square by adding to both members the square of $\frac{4ab}{2\times 2a}$ or b. (If the coefficient of x^2 is a perfect square, the trinomial square may be completed by adding *to both members the square of the quotient obtained by dividing the coefficient of x by twice the square root of the coefficient of x^2.* § 229.)

Then, $4a^2x^2+4abx+b^2=b^2-4ac.$

Extracting square roots, $2ax+b=\pm\sqrt{b^2-4ac}.$

Transposing, $2ax=-b\pm\sqrt{b^2-4ac}.$

Whence, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$

RULE: — **Reduce the equation to the form $ax^2+bx=-c$. Multiply both members by four times the coefficient of x^2, and add to each the square of the coefficient of x in the given equation.**

The only advantage of this method over the preceding is in avoiding fractions in completing the square.

1. Solve the equation $2x^2-7x=-3$.

Multiplying both members by 4×2, or 8,

$$16x^2-56x=-24.$$

Adding to both members the square of 7,

$$16x^2-56x+7^2=-24+49=25.$$

Extracting square roots, $4x-7=\pm 5$.

Then, $4x=7\pm 5=12$ or 2, and $x=3$ or $\frac{1}{2}$.

EXERCISE 102

Solve the following equations using the second method; verify all results:

1. $3m^2+10m=-3$.
2. $6t^2-13t=-6$.
3. $2r^2-15r+25=0$.
4. $5u^2+3u-2=0$.
5. $4x^2+2x-1=0$.
6. $15m^2+16m+1=0$.
7. $12x^2-11x=-2$.
8. $6x^2+11x=7$.
9. $6x^2-7x=20$.
10. $10q^2+3q=1$.

233. Solution of Affected Quadratic Equations by Formula. It follows from § 232 that, if $ax^2+bx+c=0$,

then
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \qquad (1)$$

This result may be used as a *formula* for the solution of any affected quadratic equation in the form $ax^2+bx+c=0$.

1. Solve the equation $2x^2+5x-18=0$.

Here, $a=2$, $b=5$, and $c=-18$; substituting in (1),

$$x=\frac{-5\pm\sqrt{25+144}}{4}=\frac{-5\pm 13}{4}=2 \text{ or } -\frac{9}{2}.$$

2. Solve the equation $-5x^2+14x+3=0$.

Here, $a=-5, b=14, c=3$; substituting in (1),

$$x=\frac{-14\pm\sqrt{196+60}}{-10}=\frac{-14\pm16}{-10}=-\frac{1}{5} \text{ or } 3.$$

3. Solve the equation $110x^2-21x=-1$.

Here, $a=110$, $b=-21$, $c=1$; then,

$$x=\frac{21\pm\sqrt{441-440}}{220}=\frac{21\pm1}{220}=\frac{1}{10} \text{ or } \frac{1}{11}.$$

Particular attention must be paid to the *signs* of the coefficients in making the substitution.

EXERCISE 103

Solve the following equations by formula:

1. $4x^2-7x=-3$.
2. $9u^2+22u=-8$.
3. $8t^2+10t=3$.
4. $3v^2-8v-3=0$.
5. $12=23e-5e^2$.
6. $8x^2+2x=3$.
7. $3t^2-2t=40$.
8. $m^2+7m=18$.
9. $28x^2-x-15=0$.
10. $5x^2-17x+6=0$.

234. The formula in § 233 is important in determining the nature of the roots of a quadratic equation, also in determining the relation between the roots and the coefficients in the equation.

In $ax^2+bx+c=0$,

by § 233, $$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ or } \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Call the first root r_1 and the second r_2.

I. If b^2-4ac is *positive*,

r_1 and r_2 are *real* and *unequal*.

Ex., $x^2-2x-8=0$, $b^2-4ac=4+32=+$.

Solving, $x=4$ or -2.

See Figure 1, Plate III.

II. If $b^2-4ac=0$,

r_1 and r_2 are *real* and *equal*.

Ex., $x^2-2x+1=0$, $b^2-4ac=4-4=0$.

Solving, $x=1$ or 1.

See Figure 2, Plate III.

III. If b^2-4ac is *negative*,

r_1 and r_2 are *imaginary* (§ 218).

Ex., $x^2-2x+3=0$, $b^2-4ac=4-12$.

Solving, $x=\frac{2\pm\sqrt{4-12}}{2}=1+\sqrt{-2}$ or $1-\sqrt{-2}$.

See Figure 3, Plate III.

The *intersection* of the curve with the x-axis is *imaginary*.

Imaginary roots always occur in *conjugate pairs* (§ 214).

Note that these three equations differ only in the third terms and that this difference seems to have the effect of raising or lowering the curve with respect to the x-axis.

Adding the values of r_1 and r_2, in

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a},$$

we have $$r_1+r_2=\frac{-2b}{2a}=\frac{-b}{a}.$$

Finding their product,

$$r_1r_2=\frac{b^2-(b^2-4ac)}{2a}\ (\S\ 89)=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Hence, if a quadratic equation is in the form

$$ax^2+bx+c=0,$$

the sum of the roots equals minus the coefficient of x divided by the coefficient of x^2, and the product of the roots equals the independent term divided by the coefficient of x^2.

1. Find by inspection the sum and product of the roots of

$$3x^2-7x-15=0.$$

The sum of the roots is $\frac{7}{3}$, and their product $\frac{-15}{3}$, or -5.

2. One root of the equation $6x^2+31x=-35$ is $-\frac{7}{2}$; find the other.

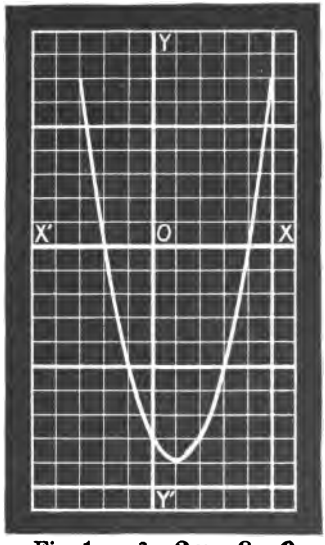


Fig. 1. $x^2-2x-8=0$

Fig. 1

x	y
0	−8
1	−9
2	−8
3	−5
4	0
5	7
−1	−5
−2	0
−3	7

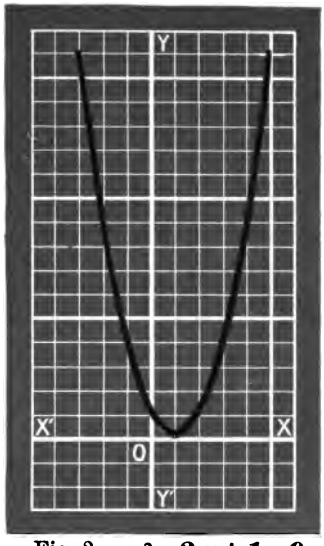


Fig. 2. $x^2-2x+1=0$

Fig. 3

x	y
0	3
1	2
2	3
3	6
4	11
−1	6
−2	11
−3	16

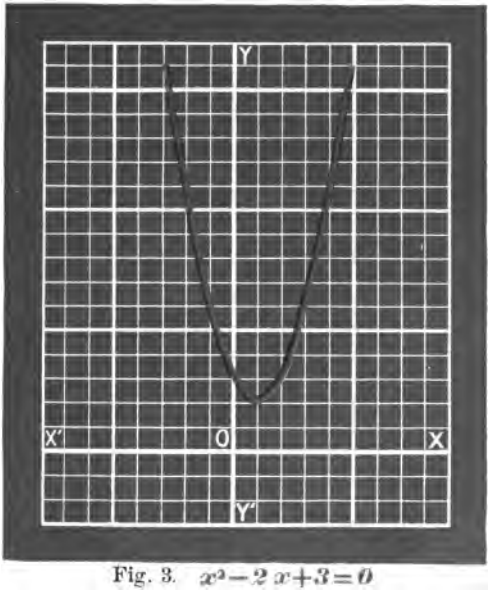


Fig. 3. $x^2-2x+3=0$

Fig. 2

x	y
0	1
1	0
2	1
3	4
4	9
5	16
−2	9
−3	16

PLATE III

The equation can be written $6x^2+31x+35=0$.

Then, the sum of the roots is $-\frac{31}{6}$.

Hence, the other root is $-\frac{31}{6}-\left(-\frac{7}{2}\right)$, or $-\frac{31}{6}+\frac{7}{2}$, or $-\frac{5}{3}$.

We may also find the other root by dividing the *product* of the roots, $\frac{35}{6}$, by $-\frac{7}{2}$.

We may find the values of certain other expressions which are symmetrical in the roots of the quadratic.

3. **If r_1 and r_2 are the roots of $ax^2+bx+c=0$, find the value of $r_1^2+r_1r_2+r_2^2$.**

We have, $r_1^2+r_1r_2+r_2^2=(r_1+r_2)^2-r_1r_2$.

But, $r_1+r_2=-\frac{b}{a}$, and $r_1r_2=\frac{c}{a}$.

Whence, $r_1^2+r_1r_2+r_2^2=\frac{b^2}{a^2}-\frac{c}{a}=\frac{b^2-ac}{a^2}$.

4. **Determine by inspection the nature of the roots of**

$$2x^2-5x-18=0.$$

Here $a=2$, $b=-5$, $c=-18$; and $b^2-4ac=25+144=169$.

Since b^2-4ac is positive, the roots are real and unequal.

Since b^2-4ac is a perfect square, both roots are rational.

EXERCISE 104

Find by inspection the nature of the roots, the sum and product of the roots, and *construct* the *graph* of each of the following 8 problems:

1. $x^2+8x+7=0$.
2. $x^2-x-20=0$.
3. $4x^2-x-5=0$.
4. $6x^2+x=0$.
5. $x^2+2x+4=0$.
6. $9x^2+6x-1=0$.
7. $9x^2+6x+1=0$.
8. $25x^2-4=0$.
9. **One root of $x^2+7x=98$ is 7; find the other.**

Note that your definitions §§ 39, 60 are involved in these examples.

10. **One root of $5x^2-17x+6=0$ is $\frac{2}{5}$; find the other.**
11. **Is 5 a root of $x^2+5x+5=0$?**

If r_1 and r_2 are the roots of $ax^2+bx+c=0$, find the values of:

12. $\dfrac{r_1^2+r_2^2}{r_1 r_2}$. 13. $\dfrac{1}{r_1}+\dfrac{1}{r_2}$. 14. $\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}$.

15. $r_1^3+r_2^3$. [Hint: $(x+y)^3+(x-y)^3$ contains but two terms.]

EXERCISE 105

Solve the following equations by the method which seems best adapted to the example under consideration, verifying each result:

(In solving any equation, we reject any solution which does not satisfy the given equation.)

1. $4t^2+3t=10$.
2. $49x^2+49x+10=0$.
3. $5h^2+12h=-4$.
4. $32v-48v^2=-3$.
5. $9m^2+6m=19$.
6. $2r^2-15r=-13$.
7. $12x^2+5x+1=0$.
8. $10-21k-10k^2=0$.
9. $\dfrac{2t+3}{8+t}-\dfrac{2t+9}{3t+4}=0$.
10. $4y+\dfrac{14-y}{y+1}=14$.
11. $\dfrac{z}{5-z}-\dfrac{5-z}{z}=\dfrac{15}{4}$.
12. $\sqrt{3+x-x^2}=2x-3$.
13. $\sqrt[3]{5s+11}=\sqrt{3s+1}+2$.
14. $\dfrac{x-2}{x+5}-\dfrac{x+4}{x-3}=-\dfrac{7}{3}$.

(Compare Ex. 19, Exercise 56.)

15. $\dfrac{x+1}{x+2}-\dfrac{x+3}{x+4}=\dfrac{8}{3}$.
16. $\dfrac{2x^2+3x-5}{2x^2-x-1}=\dfrac{3x^2+4x-1}{3x^2-2x+7}$.
17. $\dfrac{2t^2-4t-3}{2t^2-2t+3}=\dfrac{t^2-4t+2}{t^2-3t+2}$.
18. $\dfrac{1}{m^2-4}-\dfrac{1}{3(m+2)}=1+\dfrac{3}{2-m}$.
19. $\sqrt{8y+7}=\sqrt{4y+3}+\sqrt{2y+2}$.
20. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}+\dfrac{x+3}{x-3}=3$.

(Compare Ex. 14.)

21. $(x-2)(x+3)(x-4)=0$.

22. $(x-3)(2x^2+13x+20)=0$.

23. $x^2+x+1=0$.

24. $x^3=1$.

25. $\dfrac{1}{1-t^2}+\dfrac{1}{1+t}-\dfrac{1}{1-t}=-\dfrac{7}{8}$.

26. $3-\dfrac{1}{x+2}-\dfrac{3}{2(2x-3)}=\dfrac{5}{(x+2)(2x-3)}$.

27. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=\dfrac{2x+13}{x+1}$.

28. $t^3=-8$. (The roots are the three *different* cube roots of -8. Compare Ex. 24.)

29. $\dfrac{\sqrt{v}}{\sqrt{v+2}}-\dfrac{\sqrt{v+2}}{\sqrt{v}}=\dfrac{5}{6}$.

30. $\dfrac{2}{m-2}+\dfrac{2}{m-5}=\dfrac{m^2+3m-16}{m^2-7m+10}$.

31. $x^2+ax-bx-ab=0$.

We may write the equation $x^2+(a-b)x=ab$.

Multiplying both members by 4 times the coefficient of x^2,

$$4x^2+4(a-b)x=4ab.$$

Adding to both members the square of $a-b$,

$$4x^2+4(a-b)x+(a-b)^2=4ab+a^2-2ab+b^2$$
$$=a^2+2ab+b^2.$$

Extracting square root, $2x+(a-b)=\pm(a+b)$.

Or, $2x=-(a-b)\pm(a+b)$.

Then, $2x=-a+b+a+b=2b$,

or $2x=-a+b-a-b=-2a$.

Whence, $x=b$ or $-a$.

If several terms contain the same power of x, the coefficient of that power should be written in parenthesis, as shown in Ex. 1.

For the solution of literal affected quadratic equations, the methods of § 232 are usually most convenient.

The above equation can be solved more easily by the method of § 103; thus, by § 101, the equation may be written

$$(x+a)(x-b)=0.$$

Then, $x+a=0$, or $x=-a$;

and $x-b=0$, or $x=b$.

Several equations in Exercise 105 may be solved most easily by the method of § 103.

32. Solve the equation $(m-1)x^2-2m^2x=-4m^2$.

Multiplying both members by $m-1$, and adding to both the square of m^2,

$$(m-1)^2x^2-2m^2(m-1)x+m^4=-4m^3(m-1)+m^4$$
$$=m^4-4m^3+4m^2.$$

Extracting square root, $(m-1)x-m^2=\pm(m^2-2m)$.

Then, $(m-1)x=m^2+m^2-2m$ or m^2-m^2+2m
$=2m(m-1)$ or $2m$

Whence, $x=2m$ or $\frac{2m}{m-1}$.

33. $x^2-mx=m^2$; solve for x.

34. $x^2-mx=m^2$; solve for m.

Solve the following for x:

35. $x^2-2ax=-6a+9$.

36. $x^2-(a-b)x=ab$.

37. $x^2+nx+x=-n$.

38. $x^2-m^2kx+mk^2x=m^3k^3$.

39. $\sqrt{a+x}-\sqrt{2x}=\frac{2a}{\sqrt{a+x}}$.

40. $(a+b)x^2+(3a+b)x=-2a$.

41. $\sqrt{x-a}+\sqrt{2x+3a}=5a$.

Solve for t:

42. $\sqrt{5a+t}+\sqrt{5a-t}=2\sqrt{t}$.

43. $\sqrt{t^2-\sqrt{2t+1}}=t-1$.

44. $\sqrt{t+9a}+\sqrt{25a-t}=\sqrt{2t+32a}$.

45. $S=V_0t+\frac{1}{2}gt^2$.

46. $\frac{1+a}{1-at}+\frac{1-a}{1+at}=1$.

47. $\frac{t+a}{t+b}+\frac{t+b}{t+a}=\frac{5}{2}$.

(Compare Ex. 14.)

48. Solve for g: $t=\pi\sqrt{\frac{l}{g}}$.

49. Solve for s: $V=\sqrt{2gs}$.

50. Solve for n: $\frac{2n-3a}{3n+a}+\frac{3n+a}{2n-3a}=\frac{10}{3}$.

51. Solve for n: $S=\frac{n}{2}[2a+(n-1)d]$.

PROBLEMS INVOLVING QUADRATIC EQUATIONS WITH ONE UNKNOWN NUMBER

235. In solving problems which involve quadratic equations, there will usually be two values of the unknown number; only those values should be retained which satisfy the conditions of the problem.

1. **A man sold a watch for \$21, and lost as many per cent as the watch cost dollars. Find the cost of the watch.**

Let x = number of dollars the watch cost.

Then, x = the per cent of loss,

and $x \times \frac{x}{100}$, or $\frac{x^2}{100}$ = number of dollars lost.

By the conditions, $\frac{x^2}{100} = x - 21$.

Solving, $x = 30$ or 70.

Then, the cost of the watch was either \$30 or \$70; for either of these answers satisfies the conditions of the problem.

2. **A farmer bought some sheep for \$72. If he had bought 6 more for the same money, they would have cost him \$1 apiece less. How many did he buy?**

Let n = number bought.

Then, $\frac{72}{n}$ = number of dollars paid for one,

and $\frac{72}{n+6}$ = number of dollars paid for one if there had been 6 more.

By the conditions, $\frac{72}{n} = \frac{72}{n+6} + 1$.

Solving, $n = 18$ or -24.

Only the *positive* value is admissible, for the negative value does not satisfy the conditions of the problem.

Therefore, the number of sheep was 18.

If, in the enunciation of the problem, the words "6 more" had been changed to "6 fewer," and "\$1 apiece less" to "\$1 apiece more," we should have found the answer 24.

3. **If 3 times the square of the number of trees in an orchard be increased by 14, the result equals 23 times the number; find the number.**

Let x = number of trees.
By the conditions, $3x^2+14=23x$.
Solving, $x=7$ or $\frac{2}{3}$.

Only the first value of x is admissible, for the fractional value does not satisfy the conditions of the problem.

Then, the number of trees is 7.

4. If the square of the number of dollars in a man's assets equals 5 times the number increased by 150, find the number.

Let x = number of dollars in his assets.
By the conditions, $x^2=5x+150$.
Solving, $x=15$ or -10.

This means that he has assets of \$15, or liabilities of \$10.

EXERCISE 106

Verify all results.

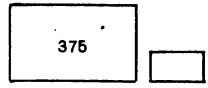
1. What number added to its reciprocal gives $2\frac{5}{12}$?

2. Divide 17 into two such parts that three times the square of the greater shall exceed twice the square of the less by 115.

3. Find three consecutive numbers such that if the square of the second number be subtracted from the sum of the squares of the first and third, the remainder will be 38.

4. The sum of two numbers is 3 and the sum of their cubes is 7; find the numbers.

5. Two rectangles have their corresponding sides in the ratio of 5 to 2. In the greater the ratio of the length to the breadth is $\frac{5}{3}$. The area of the greater is 375; find the area of the less.



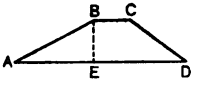
6. A farmer bought a certain number of sheep for \$300. Having lost 7, he sold the rest for \$2 a head more than they cost him, and gained \$44. How many did he sell?

7. A rectangular field is twice as long as it is wide. If 20 rods were subtracted from the length and the same amount were added to the width, the field would be square and would contain $22\frac{1}{2}$ acres. Would this change decrease or increase the area of the field?

8. A fast train's schedule from New York to Chicago is 12 miles an hour faster than a slow one, and requires 5 less hours to travel 960 miles. Find the rate of each train.

9. If the product of three consecutive numbers be divided by each of them in turn, the sum of the quotients is 107; find the numbers.

10. The area of a trapezoid is equal to the product of one-half the sum of the parallel sides and the altitude. Find the sides and altitude of trapezoid $ABCD$ in which AD is 8 feet more than BC, and EB 2 feet less than BC, the area being 55 square feet. Are there two such trapezoids?



11. A merchant sold a bill of goods for $24, making as many per cent as the goods cost dollars. Find the cost.

12. Find two numbers whose difference is 4, and the difference of whose cubes is 3088.

13. The area of a certain square field exceeds that of another square field by 1008 square yards, and the perimeter of the greater exceeds one-half that of the smaller by 120 yards. Find the side of each field.

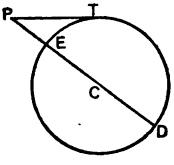
14. A and B set out at the same time from places 247 miles apart, and travel toward each other. A's rate is 9 miles an hour; and B's rate in miles an hour is less by 3 than the number of hours at the end of which they meet. Find B's rate.

15. A man buys a certain number of shares of stock, paying for each as many dollars as he buys shares. After the price has advanced one-fifth as many dollars per share as he has shares, he sells, and gains $980. How many shares did he buy?

16. The two digits of a number differ by 1; and if the square of the number be added to the square of the given number with its digits reversed, the sum is 585. Find the number.

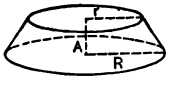
17. A merchant sold two pieces of cloth of different quality for $105, the poorer containing 28 yards. He received for the finer as many dollars a yard as there were yards in the piece; and 7 yards of the poorer sold for as much as 2 yards of the finer. Find the value of each piece.

18. In a circle with centre at C, the tangent PT is a mean proportional between the whole secant PD and the external part PE. If the tangent is 8, and the diameter ED is 12, find PE.

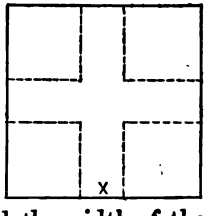


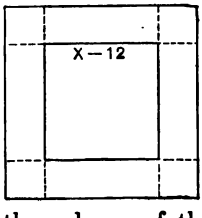
19. A and B gained in trade $2100. A's money was in the firm 15 months, and he received in principal and gain $3900. B's money, which was $5000, was in the firm 12 months. How much money did A put into the firm?

20. The formula for the volume of the frustum of a cone is $V=\frac{1}{3}\pi A(R^2+r^2+Rr)$, in which r is the radius of the upper base, R the radius of the lower base, A the altitude and V the volume. If $V=872\pi$, $r=10$ and $A=6$; find R.



21. A square garden plot containing 144 square feet has two walks of equal width intersecting at right angles to each other and to the sides of the garden. The area of the walk is one-half the area of the entire square; find the width of the walk.

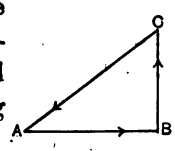




22. A square piece of tin is to be made into a rectangular box by cutting a square out of each corner and folding up the sides. The pieces cut out are 6 inches square; the volume of the box, 1944 cubic inches. How large was the sheet of tin? If a cubical box had been cut from this

sheet of tin, would its volume have been greater or less than that of the first box formed?

23. In a right-angled triangle, ABC, one side is 5 more than the other, and the hypotenuse is 5 more than the longer side. Find the dimensions. Draw diagram explaining your solutions.



24. If a body is thrown downward with an initial velocity, v_0, then the space it passes over in t seconds is found to be given by the equation

$$S=v_0t+\tfrac{1}{2}\,gt^2.$$

A stone was thrown downward with a velocity of 40 feet per second from a balloon a mile high; g is 32.15. How many seconds elapsed before the stone reached the earth?

25. In the equation $F=\dfrac{mM}{d^2}$, M and m represent the masses of any two attracting bodies, as, for instance, the earth and the moon, d represents the distance between these bodies, and F the force with which they attract each other.

If the moon had twice its present mass and were twice as far from the earth as at present, how much greater or less would the force of the earth's attraction be upon it than at present?

26. In the equation $E=\frac{1}{2}\,mv^2$, E represents the energy of a moving body, the mass of which is m and the velocity is v. Compare the energies of two bodies, one of which has twice the mass and twice the velocity of the other.

27. When a bullet is shot upward with a velocity, v, the height, S, to which it rises is given by the equation

$$v=\sqrt{2\,gS}.$$

Find with what velocity a body must be thrown upward to rise to the height of the Washington monument (555 feet). (See Problem 25, Exercise 99.)

236. Equations in Quadratic Form.

An equation is said to be in the *quadratic form* when it is expressed in three terms, two of which contain the unknown number, and *the exponent of the unknown number in one of these terms is twice its exponent in the other;* as,

$$x^6-6x^3=16;\quad x^3+x^{\frac{3}{2}}-72=0;\ \text{etc.}$$

In equations in quadratic form, the simplest method for the beginner to apply is to let some letter represent the lowest power of the unknown quantity in the given equation.

1. $x^6-6x^3=16$. Let $y=x^3$.

Then, $y^2-6y-16=0$.

Whence, $y=8$ or -2,

$x^3=8$ or -2,

$x=2$ or $-\sqrt[3]{2}$.

Verify these roots.

2. $2x+3\sqrt{x}=27$. Let $y=x^{\frac{1}{2}}$ or $\sqrt{x}$.

Then, $2y^2+3y=27$,

$(2y+9)(y-3)=0$,

$y=3$ or $-\frac{9}{2}$,

$\sqrt{x}=3$ or $-\frac{9}{2}$,

$x=9$ or $\frac{81}{4}$.

Verify these results.

3. $2s^{-8}-35s^{-4}+48=0$. Let $x=s^{-4}$.

Then, $2x^2-35x+48=0$.

Whence, $x=16$ or $\frac{3}{2}$,

$s^{-4}=16$ or $\frac{3}{2}$,

$\frac{1}{s^4}=16$ or $\frac{3}{2}$,

$s^4=\frac{1}{16}$ or $\frac{2}{3}$,

$s=\pm\frac{1}{2}$ or $\pm\sqrt[4]{\frac{2}{3}}$

EXERCISE 107

1. $z^4-29z^2=-100$.
2. $y^{-6}+19y^{-3}=216$.
3. $8t+14\sqrt{t}=15$.
4. $m^3-3m^{\frac{3}{2}}=88$.
5. $2x^{-5}+61x^{-\frac{5}{2}}-96=0$.
6. $32x^5+\frac{1}{x^5}=-33$.
7. $6m^{-\frac{1}{3}}-5m^{-\frac{1}{6}}=6$.
8. $4\sqrt[5]{x^4}+6=11\sqrt[5]{x^2}$.
9. $(2x^2-3x)^2-8(2x^2-3x)=9$.
10. $(5m+12)-5(5m+12)^{\frac{1}{2}}=-4$.

FACTORING

237. Factoring of Quadratic Expressions.

A *quadratic expression* is an expression of the form

$$ax^2+bx+c.$$

In § 94 we showed how to factor certain expressions of this form *by inspection;* we will now derive a rule for factoring any quadratic expression; we have,

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right) \\ &= a\left[x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right] \\ &= a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right] \\ &= a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right), \end{aligned}$$

by § 89.

But by § 233, the roots of $ax^2+bx+c=0$ are

$$-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a} \text{ and } -\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}.$$

Hence, to factor a quadratic expression, place it equal to zero, and solve the equation thus formed.

Then the required factors are the coefficient of x^2 in the given expression, x minus the first root, and x minus the second.

1. **Factor $6x^2+7x-3$.**

Solving the equation $6x^2+7x-3=0$, by § 233,

$$x=\frac{-7\pm\sqrt{49+72}}{12}=\frac{-7\pm 11}{12}=\frac{1}{3} \text{ or } -\frac{3}{2}.$$

Then,
$$6x^2+7x-3=6\left(x-\frac{1}{3}\right)\left(x+\frac{3}{2}\right)$$
$$=3\left(x-\frac{1}{3}\right)\times 2\left(x+\frac{3}{2}\right)=(3x-1)(2x+3).$$

2. **Factor $4+13x-12x^2$.**

Solving the equation $4+13x-12x^2=0$, by § 233,

$$x=\frac{-13\pm\sqrt{169+192}}{-24}=\frac{-13\pm 19}{-24}=-\frac{1}{4} \text{ or } \frac{4}{3}.$$

Whence,
$$4+13x-12x^2=-12\left(x+\frac{1}{4}\right)\left(x-\frac{4}{3}\right)$$
$$=4\left(x+\frac{1}{4}\right)\times\left(-3\right)\left(x-\frac{4}{3}\right)$$
$$=(1+4x)(4-3x).$$

3. **Factor $2x^2-3xy-2y^2-7x+4y+6$.**

We solve
$$2x^2-x(3y+7)-2y^2+4y+6=0.$$

By § 233,
$$x=\frac{3y+7\pm\sqrt{(3y+7)^2+16y^2-32y-48}}{4}$$
$$=\frac{3y+7\pm\sqrt{25y^2+10y+1}}{4}=\frac{3y+7\pm(5y+1)}{4}$$
$$=\frac{8y+8}{4} \text{ or } \frac{-2y+6}{4}=2y+2 \text{ or } \frac{-y+3}{2}.$$

Then,
$$2x^2-3xy-2y^2-7x+4y+6$$
$$=2[x-(2y+2)]\left[x-\frac{-y+3}{2}\right]$$
$$=(x-2y-2)(2x+y-3).$$

EXERCISE 108

Factor the following:

1. $4x^2-12x-7$.
2. x^2+x-12.
3. $25x^2-10x-11$.
4. t^2+t+1.
5. $6t^2+3t+2$.
6. $36m^2-5m-1$.

7. $20x^2-13x+1$.
8. a^2+2a+2.
9. x^4+x.
10. $6-c-2c^2$.
11. $8v^2+18v-5$.
12. a^2+4a+1.
13. $a^2+ab-6b^2+a+13b-6$.
14. $2x^2-xy-y^2+3x+3y-2$.
15. $2x^2-4xy+x-6y^2+13y-6$.
16. $6a^2+7ab-4a-3b^2+5b-2$.

238. We will now take up the factoring of expressions of the forms $x^4+ax^2y^2+y^4$, or x^4+y^4, when the factors involve surds. (Compare § 96.)

1. Factor $a^4+2a^2b^2+25b^4$.

$$\begin{aligned} a^4+2a^2b^2+25b^4 &= (a^4+10a^2b^2+25b^4)-8a^2b^2 \\ &= (a^2+5b^2)^2-(ab\sqrt{8})^2 \\ &= (a^2+5b^2+ab\sqrt{8})(a^2+5b^2-ab\sqrt{8}) \\ &= (a^2+2ab\sqrt{2}+5b^2)(a^2-2ab\sqrt{2}+5b^2). \end{aligned}$$

2. Factor x^4+1.

$$\begin{aligned} x^4+1 &= (x^4+2x^2+1)-2x^2 \\ &= (x^2+1)^2-(x\sqrt{2})^2 \\ &= (x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1). \end{aligned}$$

EXERCISE 109

In each of the following obtain two sets of factors, when this can be done without bringing in imaginary numbers:

1. x^4-7x^2+4.
2. a^4+b^4.
3. $9m^4-11m^2+1$.
4. $4a^4+6a^2+9$.
5. $36x^4-92x^2+49$.
6. $25m^4+28m^2n^2+16n^4$.

Solve the following:

7. $x^3+1=0$. (The three roots are the three different cube roots of -1.)
8. $x^4+2x^2+4=0$.
9. $x^4+8x=0$.
10. Find the three different cube roots of 27.

(Compare Ex. 24, Exercise 105.)

XV. SIMULTANEOUS QUADRATIC EQUATIONS

239. *On the use of the double signs* $\pm$ *and* $\mp$.

If two or more equations involve double signs, it will be understood that the equations can be read in two ways; first, reading all the *upper* signs together; second, reading all the *lower* signs together.

Thus, the equations $x=\pm 2$, $y=\pm 3$, can be read either

$$x=+2,\ y=+3, \text{ or } x=-2,\ y=-3.$$

Also, the equations $x=\pm 2$, $y=\mp 3$, can be read either

$$x=+2,\ y=-3, \text{ or } x=-2,\ y=+3.$$

240. Two equations of the second degree (§ 75) with two unknown numbers will generally produce, by elimination, an equation of the *fourth* degree with one unknown number.

Consider, for example, the equations

$$\begin{cases} x^2+y=a. & (1) \\ x+y^2=b. & (2) \end{cases}$$

From (1), $y=a-x^2$; substituting in (2),

$$x+a^2-2ax^2+x^4=b;$$

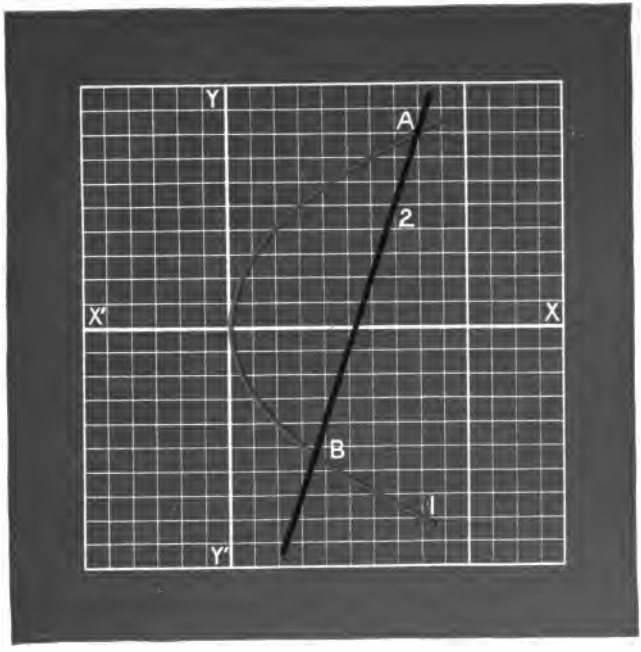
an equation of the fourth degree in x.

The methods already given are, therefore, not sufficient for the solution of every system of simultaneous quadratic equations, with two unknown numbers.

In certain cases, however, the solution may be effected. In the present work we shall consider only five simple types.

241. TYPE I. *When one equation is of the second degree, and the other of the first.*

Equations of this kind may be solved by finding one of the unknown numbers in terms of the other from the first degree equation, and substituting this value in the other equation.



(1)

$y^2 = 4x$

x	y
0	0
¼	1
1	2
2¼	3
4	4 (*A*)
¼	−1
1	−2 (*B*)
2¼	−3
4	−4

(2)

$y - 2x = -4$

x	y
0	−4
1	−2 (*B*)
2	0
3	2
4	4 (*A*)
−1	−6
−2	−8

The points A and B are the only points common to both curves. Their coördinates, (4, 4) and (1, −2), satisfy both equations and correspond to the two algebraic solutions.

In general there are two solutions of a quadratic equation and linear equation in two unknown quantities.

PLATE IV

Ex. $\begin{cases} y^2=4x. & (1) \\ y-2x=-4. & (2) \end{cases}$

From (2) $y=2x-4.$ (3)

Substituting in (1), $4x^2-16x+16=4x,$

$4x^2-20x+16=0,$

$x^2-5x+4=0,$

whence, $x=4$ or $1.$

Substituting in (3), $y=2x-4$

$=8-4,$ or $2-4$

$=4,$ or $-2.$

The solution is $x=4, y=4$; or $x=1, y=-2$. Verify by substituting in the given equations. The graphs of these equations are given in Plate IV.

242. Type II. *When the given equations are symmetrical with respect to x and y; that is, when x and y can be interchanged without changing the equation.*

Equations of this kind may be solved by combining them in such a way as to obtain the values of $x+y$ and $x-y$.

Ex. Solve the equations $\begin{cases} x^2+y^2=50. & (1) \\ xy=-7. & (2) \end{cases}$

Multiply (2) by 2, $2xy=-14.$ (3)

Add (1) and (3), $x^2+2xy+y^2=36,$ or $x+y=\pm 6.$ (4)

Subtract (3) from (1), $x^2-2xy+y^2=64,$ or $x-y=\pm 8.$ (5)

Add (4) and (5), $2x=6\pm 8,$ or $-6\pm 8.$

Whence, $x=7, -1, 1,$ or $-7.$

Subtract (5) from (4), $2y=6\mp 8,$ or $-6\mp 8.$

Whence, $y=-1, 7, -7,$ or $1.$

The solution is $x=\pm 7, y=\mp 1$; or, $x=\pm 1, y=\mp 7.$

Verify by substitution.

In subtracting ± 8 from 6, we have 6 ∓ 8, in accordance with the notation explained in § 239.

In operating with double signs, $\pm$ is changed to $\mp$, and $\mp$ to $\pm$, whenever $+$ should be changed to $-$.

The graphs of these equations will be found on Plate V. Note the symmetrical arrangement of the points of intersection.

243. Type III. *When one equation is of the third degree and the other is of the first degree.*

Certain forms of systems of first and third degree equations may be reduced to Type I or Type II by dividing one equation by the other.

Ex. $\begin{cases} x^3+y^3=18. & (1) \\ x+y=3. & (2) \end{cases}$

Dividing (1) by (2), $x^2-xy+y^2=6.$ (3)

Use Type II, squaring (2) and subtracting the result from (3),

$$-3xy=-3.$$
$$-xy=-1. \quad (4)$$

Adding (4) to (3), $x^2-2xy+y^2=5.$ (5)

$$x-y=\pm\sqrt{5}. \quad (6)$$

Solving (6) and (2) by addition and subtraction:

$$x=\frac{3+\sqrt{5}}{2}, \text{ or } \frac{3-\sqrt{5}}{2},$$

$$y=\frac{3-\sqrt{5}}{2}, \text{ or } \frac{3+\sqrt{5}}{2}.$$

The solution is $x=\frac{3+\sqrt{5}}{2}$, $y=\frac{3-\sqrt{5}}{2}$, or

$$x=\frac{3-\sqrt{5}}{2}, \quad y=\frac{3+\sqrt{5}}{2}.$$

Verify by substitution in the given equations.

244. Type IV. *When each equation is in the form*

$$ax^2+by^2=c.$$

In this case, either x^2 or y^2 can be eliminated by addition or subtraction.

1. Solve the equations $\begin{cases} 3x^2+4y^2=76. & (1) \\ 3y^2-11x^2=4. & (2) \end{cases}$

Multiply (1) by 3, $9x^2+12y^2=228.$

Multiply (2) by 4, $12y^2-44x^2=16.$

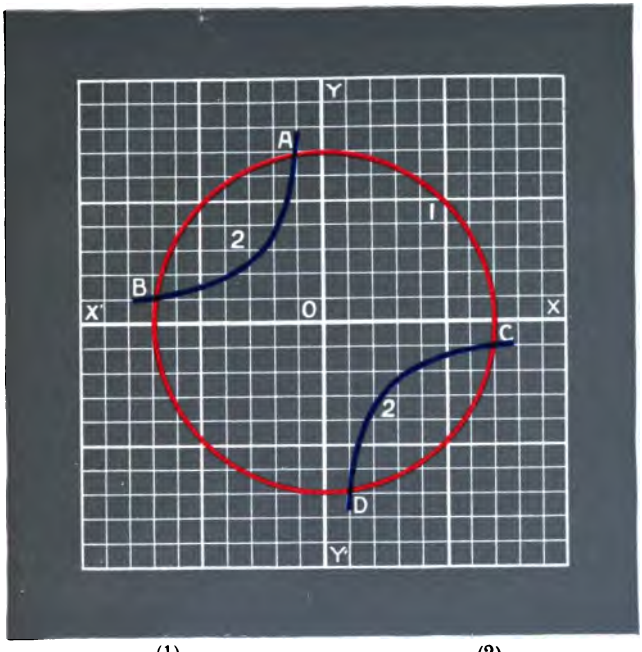
Subtracting, $53x^2=212.$

Then, $x^2=4$, and $x=\pm 2.$

Substituting $x=\pm 2$ in (1), $12+4y^2=76$, or $4y^2=64.$

Then, $y^2=16$, and $y=\pm 4.$

The solution is $x=2$, $y=\pm 4$; or, $x=-2$, $y=\pm 4.$



(1)

$x^2+y^2=50$

x	y
0	$\pm 5\sqrt{2}$
± 1	± 7 (A)
± 2	$\pm\sqrt{46}$
± 3	$\pm\sqrt{41}$
± 4	$\pm 6\sqrt{3}$
± 5	± 5
± 6	$\pm\sqrt{14}$
± 7	± 1 (C)

(2)

$xy=-7$

x	y
1	-7
2	$-7/2$
3	$-7/3$
4	$-7/4$
5	$-7/5$
6	$-7/6$
7	-1 (C)
-1	$+7$ (A)
-2	$+7/2$

etc.

In equation (1) since both x and y appear only in the second power, the double sign occurs in each substitution, so that for every pair of numerical values we obtain four points on the curve. E. g. (± 1, ± 7) gives the four points A, B, C, D. The graph of equation (2) is in two branches. (See Ex. 4, § 245.) In general two equations of the second degree in two unknowns give four solutions.

PLATE V

In this case there are four possible sets of values of x and y which satisfy the given equations:

1. $x=2, y=4.$ 3. $x=-2, y=4.$
2. $x=2, y=-4.$ 4. $x=-2, y=-4.$

It would not be correct to leave the result in the form $x=\pm 2, y=\pm 4$, for this represents only the first and fourth of the above sets of values.

The method of elimination by addition or subtraction may be used in other examples.

2. Solve the equations $\begin{cases} 3x^2-4y=47. & (1) \\ 7x^2+6y=33. & (2) \end{cases}$

Multiply (1) by 3, $9x^2-12y=141.$
Multiply (2) by 2, $14x^2+12y=66.$
Adding, $23x^2=207.$
Then, $x^2=9$, and $x=\pm 3.$
Substituting $x=\pm 3$ in (1), $27-4y=47$, and $y=-5.$

It is possible to eliminate one unknown number, in the above examples, by *substitution* (§ 157), or by *comparison* (§ 158).

245. Type V. *When each equation is of the second degree, and homogeneous; that is, when each term involving the unknown numbers is of the second degree with respect to them* (§ 59).

Certain equations of this type can be solved by the methods of §§ 242 and 244. The method of Type V should be used only when the example cannot be solved by Type II or Type IV.

Ex. Solve $\begin{cases} x^2-2xy=5. & (1) \\ x^2+y^2=29. & (2) \end{cases}$

Dividing (1) by (2), $\dfrac{x^2-2xy}{x^2+y^2}=\dfrac{5}{29},$

or $29x^2-58xy=5x^2+5y^2.$

Then, $5y^2+58xy-24x^2=0$, or $(5y-2x)(y+12x)=0.$

Solving for y, $y=\dfrac{2x}{5}$, or $-12x.$

Substituting these values in (1) we have

$$x^2-\frac{4x^2}{5}=5, \quad \text{or} \quad x^2+24x^2=5.$$

Whence, $x=\pm 5$, or $x=\pm\frac{1}{\sqrt{5}}$.

$x=\pm 5$ was obtained through

$y=\frac{2x}{5}$, whence $y=\pm 2$.

$x=\pm\frac{1}{\sqrt{5}}$ was obtained through

$y=-12x$, whence $y=\mp\frac{12}{\sqrt{5}}$.

The solution is $x=\pm 5, y=\pm 2$, or $x=\pm\frac{1}{\sqrt{5}}, y=\mp\frac{12}{\sqrt{5}}$.

EXERCISE 110

1. $\begin{cases} 3x^2+2y^2=66. \\ 9x^2+5y^2=189. \end{cases}$

2. $\begin{cases} 3x-5y^2=-116. \\ 7x+4y^2=121. \end{cases}$

3. $\begin{cases} x^3+y^3=91. \\ x+y=7. \end{cases}$

4. $\begin{cases} x^2-xy+y^2=124. \\ x+y=8. \end{cases}$

5. $\begin{cases} 4t^2+u^2=61. \\ t^2+6u^2=159. \end{cases}$

6. $\begin{cases} x^3-y^3=-117. \\ x-y=-3. \end{cases}$

7. $\begin{cases} x+y=2. \\ xy=-15. \end{cases}$ (Type II.)

8. $\begin{cases} z^2+v^2=122. \\ z+v=-10. \end{cases}$

9. $\begin{cases} x^2+k^2=26. \\ kx=5. \end{cases}$

10. $\begin{cases} u-v=4. \\ 2uv=42. \end{cases}$

11. $\begin{cases} R^2+S^2=45. \\ R-S=3. \end{cases}$

12. $\begin{cases} R^3+S^3=9. \\ R+S=3. \end{cases}$

13. $\begin{cases} L^2+LM+M^2=19. \\ L-M=1. \end{cases}$

14. $\begin{cases} xy=25. \\ x+y=10. \end{cases}$

15. $\begin{cases} xy=24. \\ 2x-y=8. \end{cases}$

16. $\begin{cases} 9t^2-5u^2=205. \\ 4t^2+9u^2=136. \end{cases}$

17. $\begin{cases} 4h^2+7k^2=32. \\ 3h^2-11k^2=-41. \end{cases}$

18. $\begin{cases} \frac{2x}{3}+\frac{3y}{2}=2. \\ \frac{3}{2x}+\frac{2}{3y}=2. \end{cases}$

19. $\begin{cases} g^2+h^2=\frac{289}{36}. \\ gh=\frac{10}{3}. \end{cases}$

20. $\begin{cases} xy=ab. \\ x-y=a-b. \end{cases}$

21. From $v=gt$ and $S=\frac{1}{2}gt^2$, find v in terms of S and g.

22. From $C=\frac{E}{R}$ and $EC=\frac{H}{t}$, find H in terms of C, R, and t.

23. From $E=FS$, $F=ma$, $S=\frac{1}{2}at^2$, and $v=at$, find E in terms of m and v.

24. $\begin{cases} x^2+y^2=25. \\ x^2-xy=4. \end{cases}$

25. $\begin{cases} 5x^2-y^2=1. \\ xy-3y^2=-10. \end{cases}$

26. $\begin{cases} p^2+pq-5q^2=25. \\ p^2+4q^2=40. \end{cases}$

27. $\begin{cases} 2x^2-xy=28 \\ x^2+2y^2=18. \end{cases}$

GRAPHS

246. 1. Consider the equation $x^2+y^2=25$.

This means that for any point on the graph, the square of the abscissa, plus the square of the ordinate, equals 25.

But the square of the abscissa of any point, plus the square of the ordinate, equals the square of the distance of the point from the origin; for the distance is the hypotenuse of a right triangle, whose other two sides are the abscissa and ordinate. Then the square of the distance from O of any point on the graph is 25; or, the distance from O of any point on the graph is 5.

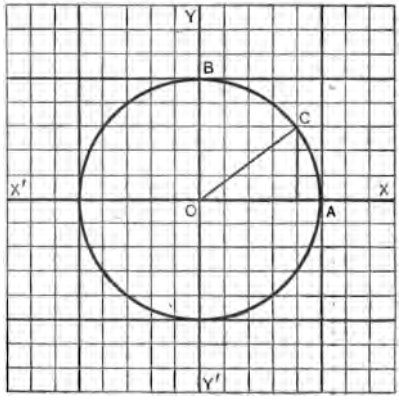


Fig. 1.

Thus, the graph is a circle of radius 5, having its centre at O.

(The graph of any equation of the form $x^2+y^2=a$ is a circle.) The graph of (1) Plate V is a circle.

2. Consider the equation $y^2=4x+4$.

If $x=0$, $y^2=4$, or $y=\pm 2$. (A, B)
If $x=1$, $y^2=8$, or $y=\pm 2\sqrt{2}$. (C, D)
If $x=-1$, $y=0$. Etc. (E)

The graph extends indefinitely to the right of YY'. (Fig. 2.)

If x is negative and <-1, y^2 is negative, and therefore y *imaginary*; then, no part of the graph lies to the left of E.

(The graph of Ex. 2 is a *parabola;* as also is the graph of any equation of the form $y^2=ax$ or $y^2=ax+b$. The graph of (1) § 241 is a parabola.)

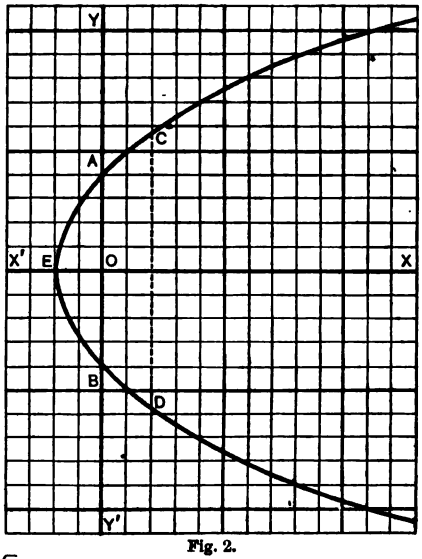


Fig. 2.

3. Consider the equation $x^2+4y^2=4$.

In this case it is convenient to first locate the points where the graph intersects the axes. (Fig. 3.)

If $y=0$, $x^2=4$,

or $x=\pm 2$. (A, A')

If $x=0$, $4y^2=4$,

or $y=\pm 1$. (B, B')

Putting $x=\pm 1$, $4y^2=3$,

$$y^2=\frac{3}{4}, \text{ or } y=\pm\frac{\sqrt{3}}{2}. \quad (C, D, C', D')$$

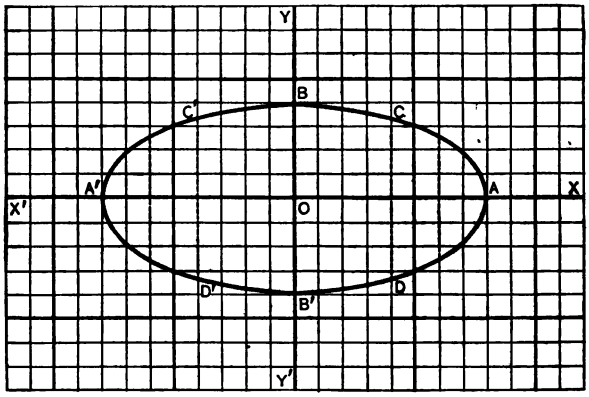


Fig. 3.

If x has any value >2, or <-2, y^2 is negative, and y imaginary; then, no part of the graph lies to the right of A, or left of A'.

If y has any value >1, or <-1, x^2 is negative, and x imaginary; then, no part of the graph lies above B, or below B'.

(The graph of Ex. 3 is an *ellipse;* as also is the graph of any equation of the form $ax^2+by^2=c$.)

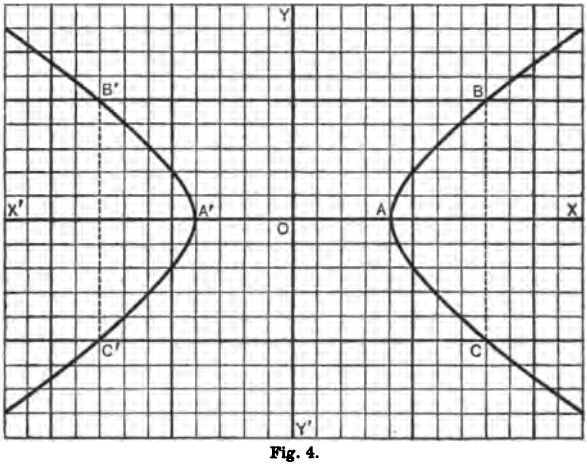


Fig. 4.

4. Consider the equation $x^2-2y^2=1$.

Here $$x^2-1=2y^2, \text{ or } y^2=\frac{x^2-1}{2}.$$

If $x=\pm 1$, $y^2=0$, or $y=0$. (A', A) (Fig. 4.)

If x has any value between 1 and -1, y^2 is negative, and y imaginary. Then, no part of the graph lies between A and A'.

If $$x=\pm 2,\ y^2=\frac{3}{2}, \text{ or } y=\pm\sqrt{\frac{3}{2}}.\quad (B, C, B', C')$$

The graph has two branches, BAC and $B'A'C'$, each of which extends to an indefinitely great distance from O.

(The graph of Ex. 4 is a *hyperbola;* as also is the graph of any equation of the form $ax^2-by^2=c$, or $xy=a$.) The graph of (2) Plate V is a hyperbola.

EXERCISE 111

Find the graphs of the following sets of equations, and in each case verify the points of intersection by comparing with the algebraic solution:

1. $\begin{cases} x^2+4y^2=4. \\ x-y=1. \end{cases}$

2. $\begin{cases} x^2-4y=-7. \\ 2x+3y=4. \end{cases}$

3. $\begin{cases} x^2+y^2=29. \\ xy=10. \end{cases}$

4. $\begin{cases} x^2+y^2=4. \\ 3x-y=8. \end{cases}$

5. $\begin{cases} y^2-3x=-3. \\ x+2y=-2. \end{cases}$

6. $\begin{cases} x^2+y^2=13. \\ 4x-9y=6. \end{cases}$

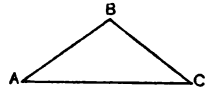
247. In solving problems which involve simultaneous equations of higher degree, only those solutions should be retained which satisfy the conditions of the problem.

EXERCISE 112

1. The sum of the squares of two numbers is 34 and their difference is one-fourth of their sum. What are the numbers?

2. The sum of the squares of two numbers is 52 and their product is 24; find the numbers.

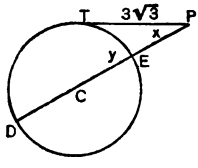
3. The sum of the sides of a triangle, ABC, is 18 inches. The sides AB and BC are equal, and the side AC is 17 less than the square of the side BC. Find the length of each side.



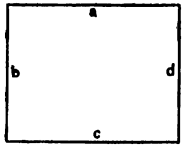
4. In a number consisting of two digits, the first digit is equal to the square of the second, and if 5 times the first digit be divided by 3 times the second, the quotient is $\frac{1}{3}$ less than twice the second digit; find the number.

5. If the length of a rectangular field were increased by 2 rods and its width diminished by 3 rods, its area would be 70 square rods; and if its length were decreased by 2 rods and its width increased by 3 rods, its area would be 110 square rods. Find the length and width.

6. A tangent TP is a mean proportional between the whole secant DP and the external segment EP. If EP equals the radius of the circle and TP is $3\sqrt{3}$, find the area of the circle.



7. The perimeter, $a+b+c+d$, of a rectangle is 36, and the area of the rectangle is 80. Find the sides.

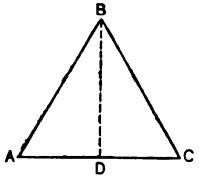


8. A farmer bought 15 cows and 20 sheep for \$720. He bought 3 more cows for \$320 than he did sheep for \$30. Find the price of each.

9. The sum of the numerator and denominator of a fraction is 7. If the numerator be diminished by 1, and the denominator be increased by 1, the product of the resulting fraction and the original fraction is $\frac{10}{9}$. Find the fraction.

10. If 7 be added to the numerator of a fraction the value of the fraction becomes 7. If the square of the denominator be subtracted from the square of the numerator the result is 7. Find the fraction.

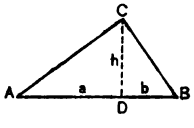
11. The area of a triangle ABC is one-half the product of the base, AC, and the altitude, DB. The area is 48 square feet. BC is 10 feet and its square is equal to the sum of the squares of BD and DC. $AD=DC$. Find AC and BD. Can more than one such triangle be drawn?



12. A triangle ABC has the angles B and C equal. The angle A is 60° more than the square of the number of degrees in the angle B. The sum of the three angles is 180°. Find the angles.

13. A travels from C to D. Two hours after he leaves C, B starts out to overtake him, traveling 3 miles per hour faster than A. Had A traveled 1 mile per hour slower, B would have overtaken him 12 miles nearer to C. Find A's rate.

14. In a triangle with a right angle at C, the altitude drawn from C to the hypotenuse is a mean proportional between the segments, a and b, of the hypotenuse. We



know also that $\overline{BC}^2 = h^2 + b^2$. If $AC = 12$, $CB = 9$, and $AB = 15$, find a, b and h.

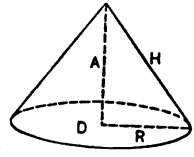
15. The sum of two numbers is to their difference as 7 is to 2. The ratio of their product is to the product of their sum and difference as 45 is to 56; find the numbers.

(Is the statement or the solution the more difficult?)

16. In a right cone, we know from geometry that

$$S = \pi RH,$$

and $$V = \tfrac{1}{3}\pi R^2 A,$$



where S = lateral surface, R = radius of base, V = volume, H = slant height, A = altitude. If $S = 60\pi$ and $H = 10$, find V. (Remember that because of the right angle at D, $H^2 = A^2 + R^2$.)

XVI. THE BINOMIAL THEOREM

POSITIVE INTEGRAL EXPONENT

249. A **Series** is a succession of terms.

A **Finite Series** is one having a limited number of terms.

An **Infinite Series** is one having an unlimited number of terms.

250. In §§ 91 and 183 we gave rules for finding the square or cube of any binomial.

The **Binomial Theorem** is a formula by means of which any power of a binomial may be expanded into a series.

251. Proof of the Binomial Theorem for a Positive Integral Exponent.

The following are obtained by actual multiplication:

$$(a+x)^2 = a^2 + 2ax + x^2;$$
$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3;$$
$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4; \text{ etc.}$$

In these results, we observe the following laws:

1. The number of terms is greater by 1 than the exponent of the binomial.

2. The exponent of a in the first term is the same as the exponent of the binomial, and decreases by 1 in each succeeding term.

3. The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

4. The coefficient of the first term is 1, and the coefficient of the second term is the exponent of the binomial.

5. If the coefficient of any term be multiplied by the exponent of a in that term, and the result divided by the exponent of x in the term increased by 1, the quotient will be the coefficient of the next following term.

252. If the laws of § 251 be assumed to hold for the expansion of $(a+x)^n$, where n is any positive integer, the exponent of a in the first term is n, in the second term $n-1$, in the third term $n-2$, in the fourth term $n-3$, etc.

The exponent of x in the second term is 1, in the third term 2, in the fourth term 3, etc.

The coefficient of the first term is 1; of the second term n.

Multiplying the coefficient of the second term, n, by $n-1$, the exponent of a in that term, and dividing the result by the exponent of x in the term increased by 1, or 2, we have $\frac{n(n-1)}{1\cdot 2}$ as the coefficient of the third term; and so on.

Then, $$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots. \quad (1)$$

Multiplying both members of (1) by $a+x$, we have

$$(a+x)^{n+1} = a^{n+1} + na^n x + \frac{n(n-1)}{1\cdot 2}a^{n-1}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-2}x^3 + \cdots$$
$$+ a^n x + na^{n-1}x^2 + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^3 + \cdots.$$

Collecting the terms which contain like powers of a and x, we have

$$(a+x)^{n+1}=a^{n+1}+(n+1)a^nx+\left[\frac{n(n-1)}{1\cdot 2}+n\right]a^{n-1}x^2$$
$$+\left[\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}+\frac{n(n-1)}{1\cdot 2}\right]a^{n-2}x^3+\cdots$$
$$=a^{n+1}+(n+1)a^nx+n\left[\frac{n-1}{2}+1\right]a^{n-1}x^2$$
$$+\frac{n(n-1)}{1\cdot 2}\left[\frac{n-2}{3}+1\right]a^{n-2}x^3+\cdots.$$

Then, $(a+x)^{n+1}=a^{n+1}+(n+1)a^nx+n\left[\frac{n+1}{2}\right]a^{n-1}x^2$
$$+\frac{n(n-1)}{1\cdot 2}\left[\frac{n+1}{3}\right]a^{n-2}x^3+\cdots$$
$$=a^{n+1}+(n+1)a^nx+\frac{(n+1)n}{1\cdot 2}a^{n-1}x^2$$
$$+\frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}a^{n-2}x^3+\cdots. \qquad (2)$$

It will be observed that this result in equation (2) is of the same *form* in $n+1$, that equation (1) is in n, and equation (2) was obtained by multiplying equation (1) by $a+x$; which proves that, if the laws of § 251 hold for any power of $a+x$ whose exponent is a positive integer, they also hold for a power whose exponent is greater by 1.

But the laws have been shown to hold for $(a+x)^4$, and hence they also hold for $(a+x)^5$; and since they hold for $(a+x)^5$, they also hold for $(a+x)^6$; and so on.

Therefore, the laws hold when the exponent is any positive integer, and equation (1) is proved for every positive integral value of n.

Equation (1) is called the *Binomial Theorem.*

In place of the denominators $1\cdot 2$, $1\cdot 2\cdot 3$, etc., it is usual to write $\underline{|2}$, $\underline{|3}$, etc.

The symbol $\underline{|n}$, read "factorial-n," signifies the product of the natural numbers from 1 to n, inclusive.

The method of proof in § 252 is known as *Mathematical Induction.*

253. Putting $a=1$ in equation (1), § 252, we have

$$(1+x)^n=1+nx+\frac{n(n-1)}{\underline{|2}}x^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^3+\cdots.$$

254. In expanding expressions by the Binomial Theorem, it is convenient to obtain the exponents and coefficients of the terms by aid of the laws of § 251.

1. **Expand $(a+x)^5$.**

The exponent of a in the first term is 5, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second, 5.

Multiplying 5, the coefficient of the second term, by 4, the exponent of a in that term, and dividing the result by the exponent of x increased by 1, or 2, we have 10 as the coefficient of the third term; and so on.

Then, $(a+x)^5=a^5+5\,a^4x+10\,a^3x^2+10\,a^2x^3+5\,ax^4+x^5$.

It will be observed that the coefficients of terms equally distant from the ends of the expansion are equal.

Thus the coefficients of the latter half of an expansion may be written out from the first half.

If the second term of the binomial is *negative*, it should be written, negative sign and all, in parentheses before applying the laws; in reducing, care must be taken to apply the principles of § 88.

2. **Expand $(1-x)^6$.**

$$\begin{aligned}(1-x)^6&=[1+(-x)]^6\\&=1^6+6\cdot1^5\cdot(-x)+15\cdot1^4\cdot(-x)^2+20\cdot1^3\cdot(-x)^3\\&\quad+15\cdot1^2\cdot(-x)^4+6\cdot1\cdot(-x)^5+(-x)^6\\&=1-6\,x+15\,x^2-20\,x^3+15\,x^4-6\,x^5+x^6.\end{aligned}$$

If the first term of the binomial is an arithmetical number, it is convenient to write the exponents at first without reduction; the result should afterwards be reduced to its simplest form.

If either term of the binomial has a coefficient or exponent other than unity, it should be written in parentheses before applying the laws.

3. Expand $(3m^2-\sqrt[3]{n})^4$.

$$(3m^2-\sqrt[3]{n})^4=[(3m^2)+(-n^{\frac{1}{3}})]^4$$
$$=(3m^2)^4+4(3m^2)^3(-n^{\frac{1}{3}})+6(3m^2)^2(-n^{\frac{1}{3}})^2$$
$$+4(3m^2)(-n^{\frac{1}{3}})^3+(-n^{\frac{1}{3}})^4$$
$$=81m^8-108m^6n^{\frac{1}{3}}+54m^4n^{\frac{2}{3}}-12m^2n+n^{\frac{4}{3}}.$$

EXERCISE 113

1. $(c+d)^4$.
2. $(x+1)^6$.
3. $(a-b)^5$.
4. $(m-k)^8$.
5. $(ab+c^2)^5$.
6. $(x+3y)^4$.
7. $(2a-b)^5$.
8. $(4h+3k)^5$.
9. $(2a^4-5b^2)^4$.
10. $(a^{-3}-2b^{\frac{1}{2}})^4$.
11. $(x^{\frac{3}{5}}+2b^{\frac{1}{4}})^5$.
12. $(1-x^2)^8$.
13. $(2a^{\frac{1}{2}}+3b^{\frac{1}{3}})^6$.
14. $(2a^{\frac{1}{2}}+3a^{-\frac{1}{2}})^6$.
15. $\left(3x^{\frac{4}{3}}-\frac{1}{2x^{\frac{4}{3}}}\right)^4$
16. $(3a^{-\frac{1}{4}}+\sqrt[6]{a})^6$.

255. *To find the rth or general term in the expansion of* $(a+x)^n$.

The following laws hold for any term in the expansion of $(a+x)^n$, in equation (1), § 252:

1. The exponent of x is less by 1 than the number of the term.

2. The exponent of a is n minus the exponent of x.

3. The last factor of the numerator is greater by 1 than the exponent of a.

4. The last factor of the denominator is the same as the exponent of x.

Therefore in the rth term, the exponent of x will be $r-1$.

The exponent of a will be $n-(r-1)$, or $n-r+1$.

The last factor of the numerator will be $n-r+2$.

The last factor of the denominator will be $r-1$.

Hence, the rth term

$$=\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{n-r+1}x^{r-1}. \qquad (1)$$

In finding any term of an expansion, it is convenient to obtain the coefficient and exponents of the terms by the above laws.

Ex. Find the 8th term of $(3a^{\frac{1}{2}}-b^{-1})^{11}$.

We have, $(3a^{\frac{1}{2}}-b^{-1})^{11}=[(3a^{\frac{1}{2}})+(-b^{-1})]^{11}$.
In this case, $n=11$, $r=8$.
The exponent of $(-b^{-1})$ is $8-1$, or 7.
The exponent of $(3a^{\frac{1}{2}})$ is $11-7$, or 4.
The first factor of the numerator is 11, and the last factor $4+1$, or 5.
The last factor of the denominator is 7.

Then, the 8th term $=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3a^{\frac{1}{2}})^4(-b^{-1})^7$

$=330(81a^2)(-b^{-7})=-26730a^2b^{-7}$.

If the second term of the binomial is negative, it should be written, sign and all, in parentheses before applying the laws.

If either term of the binomial has a coefficient or exponent other than unity, it should be written in parentheses before applying the laws.

EXERCISE 114

Find the:

1. 5th term of $(c+d)^8$.
2. 5th term of $(x+1)^9$.
3. 7th term of $(a+2b)^9$.
4. 8th term of $(a^2+b^3)^{11}$.
5. 6th term of $(a^{-\frac{1}{2}}+b^{-2})^{10}$.
6. 10th term of $\left(\sqrt[5]{m^2}-\frac{k^{\frac{3}{2}}}{2}\right)^{14}$.
7. 5th term of $(a^4-3z^{\frac{1}{3}})^{12}$.
8. 4th term of $(c^{-3}-5cd)^{15}$.
9. Middle term of $\left(3a^5+\frac{b^{\frac{1}{2}}}{2}\right)^{12}$.

THE METRIC SYSTEM

Linear Measure

The standard unit of Linear Measure in the Metric System is the **Meter**. It is determined by taking one ten-millionth part of the distance from the earth's equator to either of its poles, measured on a meridian. It is equal to 39.37 inches.

The problems in this book make use of the following subdivisions of the Meter:

10 Millimeters (mm.) = 1 Centimeter (cm.)
10 Centimeters = 1 Decimeter (dm.)
10 Decimeters = 1 Meter (m.)

MEASURES OF WEIGHT

The Gram is the unit of weight. It is equal to the weight of a cubic centimeter of distilled water at its greatest density.

The following multiples of the gram are used in problems in this book:

10 Grams (g.) = 1 Dekagram (Dg.)
10 Dekagrams = 1 Hektogram (Hg.)
10 Hektograms = 1 Kilogram (Kg.)

XVII. HINTS ON CHECKING

256. It is sometimes desirable to check a result by numerical substitutions. Any number may be substituted for the letters involved in the problem, but since all powers of 1 are 1, a substitution of 1 for a letter above the first power is not an accurate check. It is best not to use a numerical check when other means are convenient.

In *addition:* Check: Let $a=2$, $b=2$, $c=1$.

$$\begin{array}{rcl} a+2b-3c & & 2+4-3= 3 \\ -2a-b+5c & & -4-2+5=-1 \\ -3a-6b+7c & & -6-12+7=-11 \\ \underline{9a-4b-c} & & \underline{18-8-1=9} \\ 5a-9b+8c & = & 10-18+8=0 \end{array}$$

The horizontal and vertical additions being identical is a fair, not an absolute check.

In *subtraction:* Check: Let $a=b=c=1$.

$$\begin{array}{rcl} a+2b-c= & & 1+2-1= 2 \\ \underline{-4a+13b+4c=} & & \underline{-4+13+4=13} \\ 5a-11b-5c= & & 5-11-5=-11 \end{array}$$

Or add the *difference* to the *subtrahend*. The sum should be the *minuend*.

In *multiplication:* Check: Let $a=b=2$.

$$\begin{array}{ll} 2a-b & 4-2=2 \\ 3a+4b & 6+8=14 \\ \hline 6a^2-3ab & \\ \quad +8ab-4b^2 & \\ \hline 6a^2+5ab-4b^2=24+20-16=28 & \end{array}$$

If the multiplicand and multiplier are homogeneous, the product will also be homogeneous, and its degree equal to the sum of the degrees of the multiplicand and multiplier.

The Illustrative examples in § 53 are instances of the above law; thus, in Ex. 2, the multiplicand, multiplier, and product are homogeneous, and of the third, first, and fourth degrees, respectively.

The student should, when possible, apply the principles of homogeneity to test the accuracy of algebraic work.

Thus, if two homogeneous expressions be multiplied together, and the product obtained is not homogeneous, it is evident that the work is not correct.

Multiplication may be checked by using the multiplier as the multiplicand and the multiplicand as the multiplier.

In *division:*

The product of the divisor and quotient should equal the dividend. If the dividend and divisor are *homogeneous*, the quotient will be homogeneous, and its degree equal to the degree of the dividend minus the degree of the divisor.

In *factoring:*

The product of the factors should equal the given expression.

In *fractions:*

Since fractions involve the four fundamental operations, addition, subtraction, multiplication, and division, the four checks above given will suffice.

In *equations:*

Reject any root which does not satisfy the *given equations*.

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